

Rules and Reasons: Decimal Instruction for Academically Low Achieving Students

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In math classes for secondary remedial and special education students, rational number topics such as fractions and decimals tend to be taught in a highly procedural manner. The overriding emphasis is on a mastery of discrete skills and declarative knowledge. For example, when students begin learning about decimals, they are often taught to read and write decimals to the thousandths place, round decimals, and apply basic operations (i.e., addition, subtraction, multiplication, division).

Decimals are also shown to be another form of fractions. The relationship between these two topics is generally established by showing the algorithm for “dividing” fractions and thus converting them to decimals (e.g., $\frac{3}{4} = 4 \overline{)3} = .75$). Subsequent instruction using decimals involves common operations such as addition or multiplication and how, in the case of percents, to shift the decimal point two places to the left and add a percent sign. Generally, the limited special education literature on mathematics instruction offers scant details on teaching decimals beyond identifying the numbers and applying operations (e.g., Deshler, Ellis, & Lenz, 1996; Mercer, 1996; Sabornie & deBettencourt, 1997; Stein, Silbert, & Carnine, 1997).

This orientation is also reinforced by the fact that fractions and decimals, for example, have been taught this way for decades. In fact, high traditional

approaches tend to be a consistent framework for modifying decimals instructions for students with learning disabilities (see Carnine, Jitendra, & Silbert, 1997).

Another characteristic of this approach to rational numbers is that instruction follows an accelerated, highly economical format. Special educators often feel that remedial students and those with learning disabilities need to move on to more complex mathematical topics as quickly as possible. Some commercial programs for teaching fractions and decimals (e.g., Systems Impact, 1985, 1986) to academically low achieving students and students with learning disabilities allocate approximately eight to ten weeks to the two topics.

Current mathematics research calls into question the long-term effectiveness of this kind of highly procedural approach to teaching rational numbers. Three main reasons commonly appear in the literature. First, research over the last decade suggests that achieving competence in decimals takes a considerable period of time -- much longer than two to three months. Some argue that rational number concepts like decimals are some of the most difficult topics taught in elementary school (Williams, 1984) and that competence is not achieved for many average ability students until high school (Smith, 1995). This is largely due to the varied and difficult concepts that undergird decimal numbers (Behr, Harel, Post, & Lesh, 1992; Kieren, 1976).

For example, while whole numbers have a predictable order (e.g., the next number after 14 is 15), the next number in decimals is less clear (e.g., is the next number after 3.445 the number 3.446 or 3.4451?). In addition to the discrete, base 10 orientation of decimals, decimals also have continuous properties. An infinite number of decimals fall between any two decimal numbers (Hiebert, 1992).

If students are not taught the fundamental concepts or “big ideas” related to decimals, they are likely to do poorly on more complex and/or transfer tasks (e.g., seeing that $\frac{4}{5}$ of a pie diagram is equivalent to $\frac{4}{5}$ or .80 of a 10 x 10 decimal square). In other words, if all students do is to learn rules or procedures without reasons, they have little to guide them as they work, evaluate, or correct rational number problems. This observation is a consistent theme in the contemporary mathematics education literature (Hiebert & Carpenter, 1992; Leinhardt, Putnam, & Hatrup, 1992; Nesher, 1986; Skemp, 1987). Recent research (Smith, 1995) indicates that competent high school students are proficient because they have a facile understanding of core rational number concepts, and these students use an array of strategies to solve computational problems.

A second reason for reconsidering highly procedural approaches to rational numbers is that some doubt that even procedural competence by itself (e.g., the ability to divide 1.275 by .31) can be sustained without excessive and

unwarranted amounts of time devoted to drill and practice. A year-long study of third grade students who were taught to mastery levels of performance in subtraction using only procedural methods revealed that academically low achieving students tended to regress significantly in their performance over time once the teacher moved on to another math topic (Woodward, Howard, & Battle, 1997).

A related by-product of a strictly procedural orientation is an unacceptably high level of random and chronic errors (Van Lehn, 1983, 1990; Woodward & Howard, 1994) and low levels of transfer in naturalistic problem solving exercises (Bottge & Hasselbring, 1993; Hiebert et al., 1996). In the case of decimals, whole number instruction --which typically precedes the teaching of decimals -- is a major source of the misconceptions (Resnick et al., 1989).

A final concern, one directly related to what it takes to achieve high levels of procedural competence using traditional instructional methods, involves what is the most important or relevant mathematics for secondary remedial and special education students. Cawley and Parmar (1990, 1992) argue that protracted drill on math facts and computational procedures does little to prepare students for the kinds of mathematical problem solving that they will need in the world of work. A preoccupation with remedying specific math skill deficits may do little to prepare students for the kind of non-computational mathematics problem solving

activities found in adult life (Davis, 1986) such as understanding the key dimensions of a problem, gathering and organizing relevant information, estimation, and so forth.

The gap between traditional school knowledge and work knowledge is often cited in surveys of post-secondary special education students (Edgar, 1987; Halpern, 1992), and it would appear to be particularly germane in respect to mathematics at the secondary level. One could argue that much of this computational practice could be replaced with an appropriate use of calculators so that students would have more time to focus on either the conceptual foundations of a math topic or how the topic is applied in real world contexts.

The purpose of this study was to explore an alternative to the highly procedural approach to decimal instruction which is commonly found in secondary remedial track and special education classrooms. The intervention focused on a visual and conceptual introduction to decimals, followed by conceptually-based practice in the addition and subtraction of decimals. This method was contrasted with a highly procedural approach to the topic. In addition to using a clear contrast in instructional methods, a range of measures were developed to help the researchers what students in both conditions did and did not learn about decimals. In this respect, a main intent of the study was to document the effects of the two interventions on student learning.

Method

Participants

Two separate classrooms of combined eighth and ninth grade students in remedial track mathematics participated in this study. Twenty-one students were in the intervention classroom in which the instruction was conceptually based (the conceptual group). There were 23 students in the comparison or procedural class (the procedural group). The assignment of the classrooms to intervention and comparison conditions was random. In both classrooms, all of the students were at least two years below grade level in mathematics. Forty-six percent of the students received some kind of special education services in reading, writing, or mathematics. Fifteen of the 20 students were classified as learning disabled, while the remaining were classified as ADD or ADHD. Twenty-two percent of the students who participated in this study had individual educational programs (IEPs) in mathematics and were classified as learning disabled. Distributions of these students were comparable across the two classrooms. There were 11 students with disabilities in the conceptual group classroom (6 had a learning disability with IEPs for math) and 9 in the procedural group (4 had a learning disability with IEPs for math).

The Total Math portion of the Comprehensive Test of Basic Skills was used to determine the comparability of the two cohorts. A t-test was performed, and there were no significant differences between groups ($t(1,33) = .73; p = .48$).

To avoid the confounding effects of irregular attendance, students who missed five or more days during the four weeks of the intervention were excluded from the data analysis. Consequently, test scores from 17 students in the conceptual group and 18 students in the procedural group were used in the final data analysis. Three students in the conceptual group and four in the procedural group had IEPs for mathematics and were classified as learning disabled.

Materials

Conceptual group. Lessons from Mathematics in the Mind's Eye: Modeling Rationals (Bennett, Maier, & Nelson, 1988) were used as the basis for daily instruction. The program, which reflects guidelines in the 1989 NCTM Standards (National Council of Teachers of Mathematics, 1989), strongly emphasizes conceptual understanding through visual representations and physical manipulatives. Subsequent lessons also make use of these representations in order to guide students conceptually through different operations as they are applied to decimal numbers.

The physical manipulatives in the program are almost exclusively wood block rectangles, squares, or cubes (e.g., 1 x 10, 10 x 10, and 10 x 10 x 10 arrays).

Workbook drawings of these arrays are also part of the program, and they are coordinated, at times, with visual representations for fractions. For example, students are shown that pie diagrams of fractions can be equivalent to what is rendered through arrays (e.g., $\frac{2}{3}$ of a pie diagram is colored along with $\frac{2}{3}$ of a 10×10 array). The program also includes explicit links between visual representations and symbolic problems where students apply common operations to decimals (e.g., $3.21 + 44.3$).

Ten lessons from the Addison-Wesley Mathematics (Eicholz, O'Daffer, & Fleenor, 1983) were used to provide further computational practice and to give students the opportunity to solve problems using calculators. Problems from these ten lessons were used for practice on aligning decimals, rounding, estimation, and the addition and subtraction operations.

The Mathematics in the Mind's Eye curriculum was used for the basic components of the daily lesson. The teacher used the materials when he introduced a new concept on the board or with the overhead. They were instrumental when students constructed or examined visual representations individually or in small groups. Specific problems from the lesson were central to class discussions. Finally, the Mathematics in the Mind's Eye and Addison-Wesley Mathematics curricula were used as a basis for independent seatwork at the end of the period.

Procedural group. Instruction for students in the procedural group was based almost entirely on the Mastering Decimals and Percents (Systems Impact, 1986) videodisc program. This is a highly integrated program that emphasizes the application of basic operations (i.e., addition, subtraction, multiplication, division) to decimal numbers. The beginning lessons of the program teach students how to convert fractions to decimals, and the last five lessons stress the conversion of decimals to percents by shifting the decimal two places to the left and adding a percent sign (and doing the opposite for converting percents back to decimals).

The program follows well-established instructional design principles for sequencing the introduction of new concepts, teaching concepts through explicit strategies, and providing sufficient massed and distributed practice (see Carnine, Jones, & Dixon, 1994; Engelmann, Carnine, & Steely, 1992). Although the program developers frequently emphasize the importance of core concepts or “big ideas” in their research reports (Carnine et al., 1997; Kameenui & Carnine, 1998), the Mastering Decimals and Percents program’s only major concept is equivalence. By using a fraction equivalent to one, students are shown how .3 is equivalent to .30. The vast majority of the program, however, emphasizes how to apply basic operations such as addition through division to decimal numbers.

Each lesson includes voice narration, and computer graphics are used to present concepts dynamically. Guided practice problems are also presented in each lesson along with workbook practice. If students do not meet criteria on a given lesson, the program contains remedial lessons designed to insure a high level of mastery. The program is composed of 15 daily lessons.

Procedures

Pre-intervention instruction for all students. Prior to the four-week intervention, all students in this study were taught fractions in the same manner. Both classes were taught by the same teacher for six weeks, and instruction included a range of visual representations (e.g., fraction bars, number lines) to develop conceptual understanding. The teacher also used selected lessons from Addison-Wesley Mathematics to give students practice in doing computational operations with fractions. An examination of the test given at the end of the fractions unit suggested an acceptable level of criterion performance (conceptual group $mn = 81.9$, procedural group $mn = 83.5$) with no significant differences between the two groups ($t_{(1,33)} = 1.03$; $p = .32$).

Teacher selection and training for the study. The same teacher instructed the Conceptual and Procedural groups during the four week intervention. While it could have been confusing to switch from one approach to another in a typical study of different instructional methods and materials, this was less of a problem

in this study, because the Mastering Decimals and Percents videodisc program is a comprehensive intervention (Woodward & Rieth, 1997). That is, the videodisc program contains most of the daily lesson, leaving the teacher to operate the program using a remote control and to monitor students while they work at their seats. By using the same teacher, we were able to control for some of the typical “teacher effects” associated with instructional studies that involve two or more different teachers.

A researcher who had considerable experience with the Systems Impact math videodisc programs showed the participating teacher how to operate the program and conducted lessons prior to the intervention. The researcher focused this pre-intervention training on how to play the introductory portions of the lesson (e.g., modeling and guided practice), how to provide individual assistance during guided and independent seatwork practice, and how to provide remediation through the program if students have mastered a lesson.

Conceptual group instruction during the intervention. Students in the conceptual group were taught primarily through Mathematics in the Mind’s Eye: Modeling Rationals, supplemented by computational practice from the Addison-Wesley Mathematics textbook. With this group, the primary emphasis over the four weeks of the intervention was for students to develop an initial conceptual understanding of decimals. Students used different visual and physical

representations as a way of conceptualizing the discrete base ten as well as continuous dimensions of decimals (Hiebert, 1992). This instruction was intended to provide much greater depth in initial decimal concepts than these students had had in the past.

Instruction for the conceptual group also stressed the link between fractions and decimals, particularly how one representational system related to the other. For example, students were shown how $\frac{3}{4}$ in a pie diagram was equivalent to .75 in a 10 x 10 decimal square. Furthermore, the initial lessons included a focus on the equivalence between commonly used proportions (e.g., $\frac{1}{4}$ and .25, $\frac{1}{3}$ and .33). During the lessons, the teacher and students discussed how these proportional concepts appeared in everyday life (e.g., carpentry, money, cooking). The purpose of these activities was to help students understand the connection between what they previously studied in fractions and how decimals were also part of the rational number system.

Instructional content for this group also involved having students practice rounding and estimating decimal numbers, as well as adding and subtracting them. Calculators and the Addison-Wesley Mathematics textbook problems were used for this part of the instruction. It should be noted that students spent only a limited time working addition and subtraction problems by hand, and that the intervention ended before they had the opportunity to practice multiplying or

dividing decimals. Consequently, neither dividing two decimal numbers nor converting fractions into decimals through long division were included in their four weeks of instruction.

Procedural group instruction during the intervention. The teacher used the Mastering Decimals and Percents program on a daily basis. The researcher who conducted the pre-intervention training also observed in the classroom on four occasions throughout the intervention to insure fidelity of implementation. A checklist of implementation procedures was used to measure how well the teacher used the program on a daily basis. In each observation, the teacher demonstrated proficient use of the program, appropriate student monitoring, and made the correct decisions as to when to advance to the next lesson or to provide a remediation lesson.

The teacher used the Mastering Decimals and Percents remediation lessons to assure daily student mastery, and as a consequence, the total intervention for this study lasted four weeks. Students were shown a remediation lesson when one-fifth of the class did not achieve criterion performance after the regular lesson had been shown the first time. Average daily percent correct for all 15 lessons was 78 percent.

In order that students in this group had some familiarity with calculators, they were given calculators to check their work after they had completed it by

hand. This was the only instructional context in which these tools were used. However, the students were all conversant with basic calculator operations through prior mathematics instruction that had occurred earlier in the year.

An important consideration in this study was the length of the intervention. It was anticipated that a conceptual approach to decimals, one that would have included all of the basic operations, could have taken 10 to 12 weeks. This would have been far longer than the 15 lesson Mastering Decimals and Percents videodisc program, which we anticipated would take three or four weeks. Therefore, the length of the intervention for this study was determined by the time it took students in the procedural group to complete all 15 lessons in Mastering Decimals and Percents. This time limitation also helped control for threats to internal validity that could arise with much longer interventions.

Measures

Students were given two pretests prior to the intervention. The first was the Hand Computation Test, which required them to solve five decimal computation problems by hand. The problems involved adding, multiplying, and dividing decimal numbers and, in the case of the two other problems, converting proper and mixed fractions to decimals. Students took an alternative form of this measure as a posttest. Alternate form reliability for this measure was .89.

The second pretest measure was the Calculator Test, in which students used calculators to answer ten problems that were similar to those in the Hand Computation Test. In this case, there were seven problems that required adding, multiplying, or dividing decimal numbers, and three problems involved converting proper, improper, and mixed fractions to decimals. Students also took an alternative form of this measure as a posttest. Alternate form reliability for this measure was .91.

In addition to the two computation measures, a third measure, the Individual Interview, was also administered during the post test phase. For this measure, each student was given five problems covering different conceptual aspects of decimals. For example, students were asked to order decimals, write decimals based on pictorial representations, and estimate if they had enough money to buy a set of food items, all of which were shown in a dollar and cents form (e.g., 1 dozen eggs \$.89, cheese \$1.99, ham \$3.95).

During the Individual Interviews, students were asked not only to solve each problem, but also to describe what they were thinking and how they derived their answer. All individual sessions were tape recorded and transcribed for later scoring. Semi-structured individual interviews are common in decimals research (e.g., Hiebert, Wearne, & Taber, 1991; Resnick et al., 1989; Smith, 1995).

The individual student protocols were scored using a rubric that had been analytically developed based on the NCTM *Curriculum and Evaluation Standards* (National Council of Teachers of Mathematics, 1989) and related literature on innovative mathematics assessment (Lesh & Lamon, 1992). A five-point scale was used for each item, with the highest score reflecting both the numerical correctness of the student's answer and the process used to derive the answer. This kind of scoring procedure has been used in other recently-conducted mathematics research on students with learning disabilities (Woodward & Baxter, 1997; Woodward, Baxter, & Scheel, 1997).

Lastly, student performance on all three measures was examined qualitatively for specific error patterns. In addition to the many common fractions and decimals errors documented in the mathematics literature (e.g., Carpenter, Fennema, & Romberg, 1993; Hiebert & Wearne, 1986), raters identified other error patterns specific to the students' performance on the three measures. Interrater reliability for these analyses was .86.

It should be noted that logistic problems due to the school's spring standardized testing prevented immediate post testing of students following the intervention. Consequently, all three post test measures were administered ten days after the intervention concluded.

Results

Analyses of Covariance (ANCOVA) were used as a basis for evaluating performance on each of the three dependent measures. The corresponding pretests were used as covariates for the Hand Computation and Calculator Tests respectively. Each student's total test score in math on the CTBS was used as the covariate for their Individual Interview. Pre and post test descriptive statistics for all three tests are presented in Table 1.

Hand Computation Test

The ANCOVA results for the Hand Computation Test indicated near significant effects in favor of the procedural group ($F(1,32) = 3.73, p = .06$). However, mean percent correct performance shows relatively low levels of success for both groups. The procedural group averaged 34.4 percent correct compared to 22.4 percent correct for the conceptual group.

Table 2 lists by task the most common error patterns on this measure for the two groups. Students in both groups had problems aligning their columns when multiplying decimals, which usually led to incorrect answers. The conceptual group had more difficulty with the multiplication and addition tasks, at least in respect to placing the decimal in its proper location. A comparable and significant proportion of both groups had difficulty completing the division problems. That is, students in both groups either left the problem blank or tended

only to begin to answer the problem (e.g., they wrote the first number in the product and stopped).

The problems which required the students to convert fractions to decimals resulted in a different class of error patterns. For the procedural group, a sizable minority of students confused the operation; either multiplying the numerator and denominator (e.g., $5/8 = .40$) or dividing the numerator into the denominator ($3/12 = .40$). In contrast, students in the conceptual group made systematic errors by writing fractions as decimals by “reusing” the numbers in the fraction for their answer (e.g., use the 5 and the 8 in $5/8$ to derive the answer .58).

Calculator Test

For the Calculator Test, ANCOVA results showed no significant differences between the two groups ($F(1,32) = .09, p = .76$), and percent correct for both groups was relatively high. The most common error patterns for items on the Calculator Test are presented in Table 3. As indicated, students errors tended to be on problems where they were required to convert fractions to decimals. As they did with the Hand Computation Test, roughly one-fifth of the students in the procedural group multiplied the numerator and the denominator to derive their answer for these problems. Of those in the conceptual group, fewer wrote their decimals by “reusing” the numbers in the fraction.

The items involving division of decimals were problematic, particularly for the students in the conceptual group. Almost half of the students in this group inverted the numbers when they divided, whereas only 28 percent of the procedural group students did so. Finally, 17 percent of the students in the procedural group ignored the zero in the tenths column on the task where they were required to multiply two decimals.

Individual Interview

The ANCOVA for the Individual Interview measure showed significant differences favoring the conceptual group ($F(1,32) = 4.73, p = .04$). Error patterns for both groups on this measure are presented in Table 4. The most obvious error patterns appeared on the task in which students were asked to order five decimals from smallest to largest. Almost 40 percent of the procedural group used positive and negative whole number terms to describe how they ordered the decimals. For example, a student who ordered .6 as smaller than .47 gave the following reason, “11 is greater than 3, 3 is greater than .6 -- that’s like negative 6 or something -- and negative 6 is greater than negative 47 . . .” On the other hand, approximately 20 percent of the students in the conceptual group ordered decimals from smaller to larger simply based on the number of digits in each number (e.g., .6, .47, .089).

Another common error pattern emerged when students were asked to convert a visual representation of a fraction directly to a decimal (see the last example on Table 4). Half of the procedural group reused the numbers in the fraction for the decimal, and approximately 20 percent of the students in the conceptual group did so.

Discussion

A broad interpretation of these findings would suggest that relative performance on the dependent measures essentially mirrored the differences in the instructional approaches for each group. Performance on the Individual Interview, for example, significantly favored students in the conceptual group. These students were far more likely to use their understanding of key decimal concepts when answering items that required knowledge of place value and equivalence across different representational systems (e.g., the conversion of a pie diagram to a decimal number).

The higher success level for the conceptual group students on this particular task was likely the result of the specific instruction they received at the beginning of the intervention. These students spent a considerable amount of time representing decimals visually. They were shown the equivalence across, as well as within, different visual representations (e.g., the relationship between a pie chart colored to represent $\frac{1}{4}$ and a 10×10 decimal square with 25 small colored

squares; a decimal square with 75 small colored squares where students wrote $\frac{3}{4}$ as well as .75 and discussed the commonalities between the symbols and the decimal square representation).

The lack of significant differences between the groups on the Calculator Test could be explained by the nature of the items themselves (i.e., eight of the ten items involved simple, direct manipulation of decimal numbers by using basic operations) and students' level of experience with calculators. That is, in addition to the calculator practice all of the students received in class during the study, many of them had their own calculators and brought them to school on a regular basis.

Finally, there were near-statistically significant differences between groups on the Hand Computation Test, favoring the procedural group ($p = .06$), and thus suggesting a higher level of procedural competence. It is possible that the differences could have been more dramatic (and hence, statistically significant) had there been more computational problems on this measure, particularly ones that required multiplication of decimals. The number of problems on this measure was limited, however, because students in the conceptual group received very little computational practice, and when the study was designed, there were concerns about frustration and lack of motivation if students in the conceptual group were given too many problems that they would not know how to work.

Instead, the problems on the Hand Computation Test were developed to sample a range of procedural skills, especially ones taught in the videodisc program.

While this kind of comparative analysis is helpful in detailing some of the key differences in the curricula and pedagogy used in this study, it does not fully examine important issues in student learning. Careful investigations of how students learn complex concepts have become increasingly important as mathematical studies have moved away from the process-product, effective teaching orientation that characterized instructional research over a decade ago (Carpenter & Fennema, 1991; Hiebert & Wearne, 1991). Student learning is particularly important in this study because of the differences in the amount of material “covered” during the four weeks of the intervention as well as the fact that this was not the first time all of the students had been taught decimals. Like so many other topics in remedial track classes, decimals were being retaught because students had not adequately learned them the year before.

Thus, error patterns across all three measures provide a useful way of examining the different ways in which students conceptualized decimals during this study. First, and perhaps most striking, was the low level of procedural competence on Hand Computation Test. While this was anticipated for the conceptual group because of the limited practice on hand computations during the

intervention, it was unexpected for the students in the procedural group, who had studied problems like those found on this measure for four weeks.

The significant decline from average daily performance levels near 80 percent, to a mean performance of 34 percent just ten days following the intervention, suggests that economical, highly procedural approaches to decimals may not endure over time. This may be partially explained by the fact that multiplying decimals, for example, involves a considerable number of steps. If a student is wrong about one math fact, if s/he doesn't shift to the left below the line in their calculations, or if s/he puts the decimal point in the wrong place, the answer will be wrong. In this respect, the criterion for correct performance is high. Nonetheless, the general tendency for procedural knowledge alone to atrophy over time is consistent with other mathematics research (Van Lehn, 1983, 1990; Woodward et al., 1997; Woodward & Howard, 1994).

What is revealing is the range of mistakes students can make when calculating decimal numbers. This is evident in the error patterns on the Hand Computation and Calculator Tests. At times, the errors seem careless. On other occasions, the errors appear to result from fundamental confusion and subtle misconceptions. For example, students in the procedural group misaligned digits when they multiplied decimals by hand and ignored zeroes when they used a calculator (see Tables 2 and 3). Some of these patterns also characterized the

performance of the students in the conceptual group. In their case, the errors reflect tendencies that are likely to have developed prior to the intervention (e.g., when they were shown how to multiply decimals during the previous year of instruction). The conceptual group did not practice multiplying decimals by hand during the four weeks of the study nor at any other time during the school year in which the study was conducted.

A seemingly related pattern of confusion was apparent when students were asked to divide decimals (e.g., $.09 \overline{) .087}$). On the Hand Computation Test, the majority of students in both groups either did not finish the problem or simply left it blank. An analysis of performance on this measure alone might suggest that these students either don't know how to perform the operation or they do not persist in their efforts when they are faced with complex tasks. A lack of persistence is commonly attributed to students with learning disabilities (Chard & Kameenui, 1995; Kolligian & Sternberg, 1987).

Yet when these students were asked to solve a similar problem using a calculator (i.e., $.008 \overline{) .0974}$), a different pattern of errors emerged. Almost half of the conceptual group students and over one quarter of the procedural group inverted the dividend and divisor when they divided the decimal. This subtle misconception is more than just carelessness or confusion. It is understandable in

light of the different ways in which division is represented symbolically when students learn fractions and decimals.

Interviews with students in the study as well as those who participated in the alternate form reliability testing suggested that this inversion error could be attributed to earlier demonstrations of how to convert fractions into decimals (e.g., $\frac{5}{8} = 8 \overline{)5}$) as well as the close visual similarity at times between division and fractional representations (e.g., $.008 \overline{) .0974}$ and $.008 / .0974$). These different symbols for division all overlap semantically to the point that many students in this study tended to disregard the subtle but important differences in operations associated with each symbol. Visual and semantic complexities were apparent when students were asked to explain the inversion of the dividend and divisor error. Many of them casually showed the interviewer how they would divide the decimals (following the inversion error) and then explained, “I’m dividing because decimals are the same as fractions.”

Algorithmic confusion was even more apparent on the Hand Computation Test where students were asked to convert fractions to decimals. A significant number of students in the procedural group either multiplied the numerator and denominator or inverted the two numbers and divided. This tendency to multiply the numerator and denominator was also apparent on the Calculator Test.

Finally, many students, particularly those in the conceptual group, followed a “problem simplification strategy” when asked to convert fractions. That is, they simply reused numbers from the fraction to create the decimal. A core group of students in this group followed this same strategy across all three measures. These strategies are closely related to what Resnick et al. (1989) refer to as “encoding errors.” They are the by-products of the students’ extensive instructional experience with whole numbers prior to instruction in rational numbers. Because of the limited conceptions of numbers that comes from study whole numbers of a long period of time, students may focus on the numerator or denominator in a fraction as a whole number and fail to see the fraction as a ratio of two whole numbers and/or as an expression of a rational number.

This strategy also appeared on the Individual Interview for many students in the procedural group. Half of these students reused the numerator and/or denominator from the fraction when transforming a pictorial representation of a fraction to a decimal.

The fact that this error pattern occurred under these conditions and not on the two previous measures is also indicative of the way students “compartmentalize” their knowledge of decimals. That is, they are prone to using problem solving strategies inflexibly, or in a highly context specific manner (Lesh, Behr, & Post, 1987; Silver, 1986). Therefore, these students multiply a

numerator and denominator in one instance (e.g., when fractions are presented symbolically) and reuse the numbers in another context without detecting their inconsistencies.

It should be noted that we could not find consistent differences between the students with learning disabilities who participated in this study and the other remedial students. This may have been due to the small sample size of students with learning disabilities in the study or the stage of learning. That is, differences between students with learning disabilities and other students may be apparent as the latter students achieve higher levels of competence in decimals. This pattern of differential performance as the majority of students reach higher levels of competence is apparent in our past work in subtraction (Woodward et al., 1997; Woodward & Howard, 1994).

The results of this study, at least as they pertain to a student learning perspective, are limited given the scant research to date that focuses on remedial students and students with learning disabilities. More research with this population of students is needed to confirm and extend the patterns of errors described above. We would also recommend the continued use of semi-structured interviews in this kind of research, as these methods tend to confirm patterns that are only partially apparent from the paper and pencil measures.

A second limitation of the study is its relatively short duration. As so many researchers (Behr et al., 1992; Hiebert et al., 1991; Smith, 1995), have suggested, learning rational numbers takes a considerable amount of time. The intervention phase of this study lasted only four weeks, and was constrained by the time it took the procedural group to complete the prescribed 15 lessons in the videodisc program. Yet much longer studies have their own design problems, particularly in respect to internal validity concerns such as historical effects and sample mortality. These concerns are not easy to resolve if experimental or quasi-experimental methods are used for the time it might take for remedial students and students with learning disabilities to achieve much higher levels of conceptual understanding in rational numbers.

Concluding Remarks

One main finding from this study is that the results are consistent with past research on decimals instruction. It is likely to take much more time to teach rational number topics like decimals adequately than many educators anticipate. Furthermore, as Hiebert et al. (1991) suggest in their study of decimals, student learning is not likely to follow a smooth, linear path toward increasing competence. There will be occasions when students regress even when the instruction has a highly conceptual orientation. One reason for this is likely to be the intrusive effects of whole number concepts on a student's attempt to

understand decimals, particularly in the beginning phases of decimals instruction (see Resnick et al., 1989).

How to facilitate increased understanding for a student population that is prone to misconceptions and slower progress than what is typical for average ability students is a complex pedagogical and instructional design issue. Most certainly, some of the design principles used to create the videodisc program used by the comparison group can be helpful. The authors of these kinds of programs acknowledge the importance of “big ideas” (Carnine et al., 1997; Kameenui & Carnine, 1998), and this dimension needs to be a consistent feature of rational number instruction. The limited demonstration of equivalence in the videodisc program used in this study is an important first step, but it is insufficient by itself. This is apparent from the recent body of work on fractions and decimals (see Behr et al., 1992; Carpenter et al., 1993; Hiebert, 1992; Kieren, 1976; Smith, 1995). The videodisc authors’ concern for careful sequencing and attention to examples, massed and distributed practice, and feedback would also help mitigate the kinds of misconceptions found in this study. It might be added that these principles have broad support in the cognitive literature on instruction (Woodward, 1991).

Yet current mathematics research also indicates that the pedagogy used to teach rational numbers has to be much more than didactic instruction. In fact, Smith’s (1995) study of high school students who are competent in rational

numbers indicates that a significant reason for their success is that these students have gone well beyond what is presented in textbooks. He recommends active classroom discussions of different strategies for solving rational number problems as an important way of developing a deeper understanding of basic concepts. This strategy, along with the role of multiple representations, are common recommendations in the contemporary literature on mathematics instruction (Ball, 1993; Hiebert & Carpenter, 1992; Lampert, 1990; Silver, 1986).

Finally, the special condition of secondary remedial students and students with learning disabilities cannot be ignored. Many of these students have limited time remaining to study mathematics. As Cawley and Parmar (1992) have noted, these students need more problem solving experiences where rational number concepts, for example, can be applied to realistic contexts. This suggestion is consistent with what Kieren (1993) proposes when he recommends that students learn difficult concepts like rational numbers recursively. That is, students need to learn these concepts from symbolic, visual, and intuitive frameworks.

Instruction that addresses intuitive understanding either builds on informal knowledge, or couches it in mental models which are more readily available to students. Anchored instruction (Bottge & Hasselbring, 1993; Cognition and Technology Group at Vanderbilt University, 1993), which has become increasingly prominent in special education, attempts to build intuitive knowledge

by situating students in rich, naturalistic contexts for problem solving, such as ones involving hypothetical employment settings.

While not addressed explicitly, a more recursive style of instruction for academically low-achieving secondary students may be a well-suited addition to the conceptually-based instructional program described in this study. This model may not only enhance conceptual understanding by enabling students to learn a concept through multiple representation systems, but it would be a useful way to prepare these students for the mathematics they will need as they make the transition from school to work.

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