Making Reformed Based Mathematics Work for Academically Low Achieving
Middle School Students

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For many years, special educators have perceived competence in mathematics to be fluency in math facts and computational procedures as well as the ability to solve problems quickly and accurately. This image of basic skills mastery is alluring because these targeted outcomes are relatively straightforward. They lend themselves to a hierarchical instruction, with each step in the sequence taught to mastery. This perspective, however, is a great distance from what the National Research Council’s *Adding It Up* (Kilpatrick, Swafford, & Findell, 2001) recently described as mathematical proficiency: (a) conceptual understanding; (b) procedural fluency; (c) strategic competence, the ability to formulate and represent problems; (d) adaptive reasoning, the capacity for logical thought, explanation and justification; and (e) productive disposition, the belief that mathematics makes sense and is useful. These dimensions of proficiency make for an ambitious agenda for moving American students to much higher levels of mathematics achievement.

Initially, many special educators were skeptical of the mathematics reform of the early 1990s, labeling it elitist and characterizing it as recycled efforts from the new math of the 1960s (see Woodward & Montague, 2002). However, continued efforts to reform math in the US based on ongoing research, new curricula as well as the impetus of new NCTM Standards (National Council of Teachers of Mathematics, 2000) suggest that mathematics reform is far from an educational fad or pendulum swing that will eventually return to basic skills instruction. In fact, arguments today are less about whether standards should guide math education and more about which standards should be adopted and how they should be implemented (Loucks-Horsley, Love, Stiles, Mundry, & Hewson, 2003).

The gap between typical special education practice and the current state of mathematics reform can be explained on at least four grounds. The first stems from fundamental assumptions about instructional interventions in special education. The professional literature as well as the way in which graduate students are prepared in special education tend to focus on broad based
instructional interventions, particularly instructional and classroom organization strategies over indepth knowledge of a specific discipline. Consequently, the research in the field is filled with studies on curriculum based measurement, peer tutoring, direct instruction, strategy based instruction, and the like. For many researchers, the specific content and structure of a discipline (e.g., math, science, social studies) is secondary to these broader interventions. The way special educators tend to think about instruction also tends to place a great emphasis on the skills in a discipline. This is one reason why math (a discipline) is easily confused skills instruction (e.g., phonics).

A second and related reason is historical. In the late 1970s and early 1980s, prominent research in remedial, special education and general education research coincided. Effective teaching research and direct instruction shared many common principles (Becker, 1977; Brophy & Good, 1986). In an effort to help students develop mastery of basic skills, teachers were encouraged to move through a lesson at a brisk pace, model new procedures, ask low level questions that allowed immediate feedback, and provide extended opportunities for students to complete carefully supervised seatwork or homework. Many of these practices were called into question two decades later through mathematics research in this country (see Grouws, 1992; Putnam, Lampert, & Peterson, 1990) and international comparison studies (e.g., the Third International Mathematics and Science Study or TIMSS).

A third major reason for the distance between the common special education vision of mathematics instruction and the wider view held in the mathematics education community is a deep distrust of constructivism, which is commonly associated with math reform. Some special educators present oversimplified accounts of constructivism, suggesting that it is nothing more than discovery learning (Carnine 2000). Geary (2001) provides a good example of the confusion around constructivism in his recent essay on the evolutionary dimensions of learning. He seems
to dismiss constructivism as opposed to the kind of skills instruction needed in math. At the same time, he acknowledges the importance of classroom discussions, particularly opportunities for children to explain their thinking to others. The latter practice is fully consistent with social constructivism. Some of this confusion undoubtedly stems from the asymmetry between constructivism as a theory of learning and what has appeared over time as a diverse set of instructional principles. There are multiple visions of constructivist practice and thus, it is a mistake to characterize its instructional methods in one way. The concept of scaffolding, which is central to constructivist practice, suggests that constructivism is much more than open ended, discovery learning. Nor is it the case that constructivism prevents teachers from focusing on skills. As the section below on number sense suggests, skills instruction needs to be thoughtfully integrated with more meaningful activities.

A final reason for the gap between current math reform and special education practice has to do with learner characteristics, particularly children who are broadly labeled as having learning disabilities. Most mathematics intervention research is conducted on this population of students. Unfortunately, this special education category is notoriously ill-defined. Some would claim that the vast majority of students with learning disabilities are misclassified, and that their main difficulties are reading problems (Lyon, et al., 2001). Others in the field intimate that a distinct subset of LD students suffers from dyscalculia, and that these students have a difficult time acquiring basic skills like math facts if they can learn them at all (Geary, 1994, 2004). Still others question the construct of learning disabilities from a philosophical and sociological standpoint (Reid & Valle, 2004). Their accounts suggest that social and cultural factors significantly outweigh neurological impairments as an explanation for why students end up (or should end up) being labeled LD.
Even more problems with the LD category can be found by examining the math intervention literature. When studies are conducted in naturalistic settings, the accessible samples often possess a wide range of characteristics because cooperating districts, not researchers, qualify these students as learning disabled. As a consequence, many students who, under other conditions would be labeled as mildly developmentally delayed or behavior disordered, are classified as LD. The sum effect of all of this is that it is difficult to link one best practice to such a heterogeneous population of students. Put in a mathematical context, some students classified as learning disabled would be best served with consumer math and daily living skills rather than a continuum of instruction that leads them toward increasingly ambitious mathematics.

Aligning Mathematics Instruction for Students with LD with Mathematical Proficiency

The remainder of this chapter will describe new ways of conceptualizing math instruction for many students with LD. The different sections draw on emerging directions in the field of special education where researchers are attempting to rethink or broaden the instructional experiences for these students. It is hoped that this perspective will move students toward increased mathematical proficiency as articulated in Adding It Up (Kilpatrick et al., 2001). Finally, as suggested above, LD as they exist in schools (versus idealized accounts in the professional literature) are markedly heterogeneous. Thus, what is presented may not be well suited to all students in the category of LD.

Developing Number Sense

Special educators have recently become interested in the development of number sense as a foundation for mathematics understanding. Gersten and Chard (2000) suggest that intense remediation in number sense will better prepare primary grade students at risk for special education to be competent in the skills needed to start learning topics like addition and subtraction. Their argument draws on 20 years of mathematics research, beginning with Gallistel
and Gelman’s (1986) seminal work with 3 to 5 year olds. This work established a clear progression in the ability to make a one-to-one correspondence between counting words and objects to the principle of order irrelevance (i.e., the ability to count objects in any order). Subsequent research by Siegler (1996) and others underscores the complexity of this kind of development. In fact, Siegler documents how students can demonstrate a range of mathematical strategies in the early primary grades, and the choice of a strategy often depends on the task. This line of research has culminated in Griffin, Case, and Sigler’s (1994) Right Start program, which presents counting skills in the context of number line games.

Promising as early number sense interventions might be, they can be problematic on two accounts. First, there is the danger of transforming number sense activities into direct instruction drills. Current efforts in early reading make the notion of drill on basic skills plausible. However, there is a vast difference between the kind of convergent, drill based instruction that might lead a student to identify the letter /f/ and say its sound and the kind of understanding associated with a number like 4. Figure 1 below shows the range of concepts associated with this number, virtually all of which can be learned in the primary grades.

Clearly, many of these representations are learned over time as students move through the primary grades, and as Fuson (1988) pointed out, number lines are potentially confusing for young learners because of the number of tick marks (i.e., there are 5 tick marks that are used to represent the distance of 4).

The second problem is the implication that number sense development is a primary grade phenomenon. Liping Ma’s (1999) now classic interviews with American and Chinese teachers
present tacit examples of how number sense develops throughout the elementary grades.

Arguably, number sense develops concurrently across the entire mathematics experience. As one example of Ma’s research, American teachers tend to show students only one method for solving subtraction problems like the one below.

\[
\begin{array}{r}
5 \\
2 \\
\hline
3 \\
9 \\
\hline
2 \\
3
\end{array}
\]

In contrast, Chinese teachers show students multiple ways of solving the problem. Each method turns on the ability to decompose numbers by place value and work efficiently with facts. For example, students can learn how to work “from the top” by decomposing 62 into 50 + 12, and 39 into 30 + 9. Once this is accomplished, students can subtract 50 – 30 and 12 – 9. This yields 20 and 3 or 23. An alternative method would be to decompose 62 into 50 + 10 + 2 and 39 into 30 + 9. Once broken up in this fashion, students can learn to subtract in this fashion: 50 – 30 and 10 – 9. Students then add 20 + 1 + 2 to get an answer of 23.

To be sure, it is unclear at this point how many methods are suited to academically low achieving students in the elementary grades. There is little reason to believe that exposing all students to multiple (or invented) algorithms will insure success for everyone. Figure 2 presents a more conservative array of methods for helping students think about addition.

At the top, students learn the traditional algorithm and contrast this understanding with an expanded algorithm. The advantage of the former is that it is efficient. The advantage of the latter is that students explicitly see two key concepts: place value and regrouping. This
understanding is enhanced through the use of number blocks, which also help students understand the regrouping process.

The next portion of Figure 2 shows how the same addition problem can be presented horizontally and the numbers decomposed by place value. It should be apparent that this kind of presentation informally introduces students to the commutative property and, more generally, the horizontal structure of mathematics found in middle school onward. In all likelihood, this kind of presentation is best suited to academically low achieving students in the intermediate or middle school grades.

Finally, Figure 2 shows the value of approximations as a core dimension of number sense. Students need only learn a limited number of strategies for rounding and then approximating answers to problems. The first strategy encourages students to round up or down to the nearest decade or hundred if applicable. An extension of this strategy is to compensate with problems like 44 + 24. In other words, both call for rounding down. However, the effects of rounding both numbers down can be contrasted with rounding one of the numbers down and the other one up. Consequently, the compensation approach yields a closer approximation than rounding both numbers down. The quarters strategy builds off of the concept of money and has students round to 25, 50, or 75. As students are introduced to different strategies over time, opportunities for discussion arise regarding which strategy is best suited to a particular context. Moreover, it is critical that number sense arise from informal as well as formal contexts.

Some math researchers feel that students are most likely to expand their understanding of numbers when it is part of a classroom “environment” Greeno (1991). It is equally important that number sense is an agenda that develops concurrently with every mathematical topic. For example, being able to use benchmarks to talk about decimal numbers such as .3147 as about 1/3 is important number sense as well as connections between two types of rational numbers.
Identifying compatible numbers and commuting to simplify the following algebraic expression also involves number sense.

\[ 17 + 6x + 9 + x + 3 + 4x \rightarrow 17 + 3 + 9 + 6x + 4x + x \rightarrow 29 + 11x \]

These kinds of activities, which occur across the grade levels as students learn mathematics, help develop conceptual understanding. Naturally, this needs to be coupled with the development of procedural fluency, particularly in math facts and extended facts (e.g., 4 + 5 = 9, 40 + 50 = 90). The emphasis on strategies for learning facts as well as the use of approximation in solving mathematical problems also fosters the kind of strategic competence described in *Adding It Up* (Kilpatrick et al., 2001).

**Conceptual Understanding**

A central theme in mathematics education since the mid 1980s has been the importance of conceptual understanding to mathematical competence. A series of influential texts on mathematical concepts written over the last 15 years emphasize the importance of conceptual understanding as a guide to procedural fluency (e.g., Carpenter, Fennema, & Romberg, 1993; Hiebert, 1986; Leinhardt, Putnam, & Hattrup, 1992; Ma, 1999). The importance of conceptual understanding was reinforced recently in reflections on the results of the TIMSS. Schmidt, one of the senior authors of the TIMSS research emphasized the importance of conceptual understanding as a common thread in successful mathematics instruction around the world (Math Projects Journal, 2002; Schmidt, McKnight, & Raizen, 1997).

The role of conceptual understanding is certainly not restricted to the manipulation of numbers. Other core topics such as measurement and geometry are excellent venues for exploring concepts and problem solving. Furthermore, math research indicates that elementary and secondary students are prone to significant misconceptions in topic areas such as geometry (Barrett, Clements, Klanderman, Pennisi, Polaki, April, 2001; Toumasis, 1994). Regrettably, there
is little research in the special education literature on topics such as geometry. Carnine (1997) offers a rare description of geometry instruction for students with learning disabilities. Using what he calls an instructional design perspective, he describes how an understanding of volume formulas for common three-dimensional objects can be reduced to variations on base • height. This organization enables students to see “the sameness” between the objects that is otherwise masked by a disparate set of formulas. For example, the volume formula for a cylinder is base • height. However, the formula for the volume of a sphere is typically written as \( \frac{4}{3} \pi r^3 \). Carnine shows an alternative formula for a sphere written as base • \( \frac{2}{3} \) • height. The alternative formula uses the area of a circle as a base and a perpendicular diameter that intersects this base as the height. However, it is not clear if the students ever compare the traditional formula with its alternative to see how they are equivalent.

The potential advantage of this kind of treatment is that it does more than draw attention to common role of base and height in volume formulas. If these concepts are presented with physical models, students can explore the relationship between two and three-dimensional objects. Carnine does not indicate if this is a complementary activity, and one is left wondering if the ultimate purpose of this sameness analysis is to facilitate an easier method for students to memorizing formulas for various pyramids, prisms, and a sphere.

An alternative and much richer framework for thinking about geometry concepts comes from the work of van Hiele (Fuys, Geddes, & Tischler, 1988). Rather than focus on vocabulary and memorizing formulas, the van Hiele system places considerable emphasis the role of observation and problem solving in the early stages of learning about geometric objects. The system then gradually builds toward the more formal geometry that is studied at the high school and college level. Figure 3 below presents the four basic levels of van Hiele’s system. Fuys et al. argue that American textbooks frequently mix these levels and are largely preoccupied with Level
3. Consequently, many students do not receive an adequate foundation in geometric thinking and fail to achieve complex levels of understanding. This is less of an issue for students with disabilities because of the focus on computations and one or two-step word problems in the research literature and that most day-to-day instructional settings largely ignore geometry.

A system like van Hiele’s can be a constructive way of thinking about geometry instruction for students with I.D. Examining different shapes and then grouping them into different categories (e.g., objects with straight sides, objects with at least one acute angle, objects with convex surfaces) can be a Level 0 task that promote logical thinking. Finding lines of symmetry on cut-out objects and creating tessellations as ways to slide, flip, and turn objects are possible Level 1 activities.

As students move toward higher levels of geometric understanding, they have the opportunity to explore the underpinnings of many of the formulas used for two and three dimension objects. This aspect of the van Hiele system is particularly crucial, because it enables students to visualize properties of objects and the rules or formulas used to describe these objects. For example, triangles can play a significant role in many two dimensional shapes. A Level 2 starting point for investigating different types of triangles is to have students use metric rulers and protractors to measure the sides and angles of various triangles and then discuss their results. Examples of the kinds of triangles that could be used for this kind of instruction are shown in Figure 4.
A discussion of the properties of the six different triangles should yield the observation that triangles A and C have equal sides and equal angles (i.e., equilateral triangles), triangles B and E have two sides and two angles that are the same measure (i.e., isosceles triangles), and triangles D and F have unequal sides and angles (i.e., scalene triangles). The potential benefit of this approach is that it turns the rote memorization task of simply identifying three different types of triangles into one where students need to create classifications based on common features.

An investigation of triangles can be extended into area formulas for many two dimensional objects. For example, it is relatively easy to communicate the formula for the area of a square or rectangle through arrays as shown at the top of Figure 5. Students can readily see that the area formula for these two objects is length • width or to use consistent language for two dimensional objects, base • height.

Students can see how other area formulas were derived through a series of Level 1 activities. For example, bisecting a rectangle as shown in Figure 5 into two congruent triangles helps students see that the area triangle is ½ base • height. Manipulating the triangular portion of a parallelogram (e.g., cutting and then reassembling) helps students see that the area of a parallelogram is the same as that of a rectangle.

A similar logic applies to the formula for the area of a trapezoid. Dividing the trapezoid into two triangles creates the basis for seeing how area formula for the trapezoid is derived. Seeing the traditional formula requires one more step where the ½ is factored as a coefficient.

Triangles can play a role in helping students understand how to derive the area of any polygon. Figure 5 also shows how an irregular polygon like the pentagon can be divided into
triangles in order to find its area. Regular polygons such as the hexagon allow students to use problem solving strategies such as making a simpler version of the problem in order to find the area. In this case, the student might subdivide the hexagon into six equilateral triangles, measure the base and height of each one, and then multiply that area by six.

Finally, well-structured activities that allow students to investigate properties of circles enable students to understand the concept of pi as well as the formula for the area of a circle. As Figure 5 indicates, students can measure the diameter and circumference of a cylinder and compare the two measurements. Diameter divides circumference 3.14 times. Taking this kind of exploration a step further, if circles are systematically divided into small wedges or what appear to be small triangles and reassembled into a parallelogram, students can get an approximate sense of how the area formula $\pi r^2$ is derived.

This kind of foundation in two dimensional objects lays the groundwork for studying properties of three dimensional objects. Using the van Hiele system as a guide, an initial Level 0 activity would be to compare properties of prisms and pyramids. Figure 6 shows that these comparisons entail considerations of cylinders and cones as respective members of the different categories of prisms and pyramids.

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Figure 6 indicates how students can investigate the relationship between two and three dimensional objects, specifically prisms. For example, triangular prisms are comprised of a stack of triangles in the same way that a cylinder is a stack of circles. Pyramids, particularly cones, present a different kind of conceptual challenge simply because rotating a right triangle 360 degrees will be inadequate for understanding volume formula for a cone of $1/3$ base • height.
Like the volume of a sphere, there are no easy ways to link the formula with a robust visualization of how the formula is derived. However, a tactile way to prove the cone formula is simply to pour the contents (i.e., the volume) of a cone into a cylinder of the same diameter and height. As Figure 6 indicates, three cones fill the cylinder, hence the volume of the cone is 1/3 that of the cylinder.

These examples of geometry thinking based on the van Hiele system indicate a shift in what is valued knowledge in mathematics instruction for students with learning disabilities. Hiebert (1999) has argued that the current shift in mathematics instruction in the US – and arguably, at a global level (see Journal of Learning Disabilities, January/February 2004) – has to do with an emerging consensus on what students need to know in the world today and the near future. Clearly, the increasing role of technology has devalued the importance of computational fluency alone. Activities that promote conceptual understanding like the ones described above also provide opportunities for adaptive reasoning, logical thinking, and explanation as well as a productive disposition described in Adding It Up (Kilpatrick et al., 2001).

Problem Solving

For many years, problem solving for students with learning disabilities involved what some researchers have characterized as “end of the chapter” problems (Goldman, Hasselbring, and the Cognition and Technology Group at Vanderbilt, 1997; Woodward, 2004). These problems typically are found in traditional math textbooks and special education curricula. Students answer a set of one or two step word problems that provide practice on an algorithm such as multiplication that was also part of the textbook chapter. The typical strategy is to search for a key word (e.g., “more” means to add, “each or every” means multiply or divide) and use this information to directly translate the problem into its computational form.
Cognitive research conducted in the 1990s showed that this kind of instruction was not only limiting, but it was generally associated with poor problem solvers. Hegarty and her colleagues (Hegarty, Mayer, & Green, 1992; Hegarty, Mayer, & Monk, 1995) have demonstrated that students who search for key words and do not spend sufficient time representing the problem are prone to a significant number of errors when problems are presented in an inconsistent format (e.g., where the correct solution to a word problem with “more” required subtraction rather than addition).

Recent problem solving research in special education indicates an important move away from direct translation methods toward strategy based instruction, problem representation, and dialogue (Jitendra, 2002; Jitendra, DiPipi, & Perron-Jones, 2002; Montague & van Garderen, 2003; van Garderen & Montague, 2003). Work of this kind is critical because of the tendency on the part of so many students, particularly those with LD, to answer problems impulsively. These students need to learn how to persist when performing complex tasks (Kolligian & Sternberg, 1987). In fact, a striking characteristic of so many students in mathematics classes is captured by the mistaken belief that all math problems can (and should) be answered in five minutes or less (Doyle, 1988; Schoenfeld, 1988).

In an effort to assist students in developing strategic knowledge and persistence, Woodward and his colleagues (Woodward, Monroe, & Baxter, 2001) created small group contexts for problem solving. Groups consisted of no more than eight low achieving and LD students worked together in class once a week for approximately 30 minutes. This structure enabled opportunities for all students to discuss one or two complex problems. It should be emphasized that students were encouraged to solve each problem collaboratively. Hence, they could observe and contribute based on their initiative or when asked by the teacher. This arrangement was in marked contrast to settings where students are expected to answer problems individually. Finally,
the structure also facilitated the kind of fine grained scaffolding that is infeasible in whole class discussions (Stone, 1998).

Rather than memorize strategies, students were given laminated guides or “bookmarks” that prompted them through the steps of problem solving (i.e., read and re-read, find out what the problem is asking for, select and execute a strategy, and evaluate the answer). Furthermore, the bookmark listed six of the most common domain specific strategies for solving math problems (e.g., make a simpler problem, work backwards, guess and check, make a table or organized list). Pilot work for this research indicated that these were often more useful strategies than the commonly recommend strategy of “draw it.”

Figure 7 presents the problem as well as a portion of the teacher-students’ dialogue. The dialogue begins after the teacher and students had read the problem twice and talked about what the problem “was asking for.” One of the most notable features of the dialogue is the varied participation. The teacher insures that no single student or subset of students dominates all of the tasks. In addition to scaffolding student responses, the teacher often revoices student comments and, in doing so, inserts mathematical vocabulary into the discussion (O’Connor & Michaels, 1996).

Perhaps the most intriguing moment in this dialogue occurred when the teacher erred in recording student guesses on the whiteboard. Inadvertently, the teacher wrote 51 and corrected herself to write 52. The same mistake occurred on the next guess, when she wrote 50 instead of 55. Observational notes from this lesson indicated that Terrell had not made any contributions to the discussion up to this point in the lesson. Instead, he watched as others either answered a portion of the problem (e.g., Alicia’s answer to the first question) or offered guesses. It was when
the teacher made repeated errors recording student suggestions that he offered an alternative solution. A follow up interview with Terrell indicated that he thought the way the teacher was doing it was “too messy” and that a list would make it easier to find the right answer. Observations over several months of problem solving suggested that “making a list” or “make a table” were powerful strategies that could be used to solve a range of problems.

Two other observations about problem solving arise from this example. First, this dialogue occurred early in the process of small group problem solving. The goal at this point in the intervention was to create a working community where students felt comfortable sharing their ideas with others in a discussion. This kind of risk taking is not to be underestimated given how these students often opt out of whole class discussions (Baxter, Woodward, & Olson, 2001). This goal contrasts sharply with what is more typically associated with the need for individual accountability in math classrooms (Putnam & Borko, 2000). Researchers resolved this conflict between group and individual work over time by having students work a problem individually after the small group work that was similar to the one used in the discussion. Also, there were increasing occasions in whole class settings when these students worked problems by themselves or with minimal teacher prompting.

A second consideration has to do with group participation. Explicit, verbal participation in group work is often considered to be the index of engagement and learning. However, Terrell’s contribution, which occurred 15 minutes into the lesson, suggests that learning can occur through observation in well constructed settings. This point is consistent with Lave and Wenger’s (1991) notion of practical peripheral participation. There is little research on the effects of environments that promote learning through observation accompanied by scaffolded verbal interactions in the special education literature.
Implications for Instruction

These kinds of approaches to instruction for late elementary and middle school students with LD -- as well as those in remedial mathematics classes -- cannot be accomplished without substantive changes in curriculum and instructional practices. Curricular change is essential because it can authorize different methods and instructional goals. All three topical examples described above make this clear. The development of number sense requires different kinds of instructional examples and different forms of practice. New curricula are also essential because teachers are in no position to develop and then validate new approaches by themselves. Conceptual frameworks and new methods that appear in the professional literature do not readily transfer to the classroom on a sustained basis. Finally, curricular materials which are consistent with the 2000 NCTM Standards need to be adapted for struggling students. There is no reason to believe that one set of curricular materials will serve the needs of all students in a heterogeneous, middle school classroom. Our earlier work indicated that the pace of instruction as well as the complexity or cognitive load of the materials was too much for struggling students (Baxter et al., 2001; Woodward & Baxter, 1997).

These students need materials that make more explicit connections between topics (e.g., the development of number sense strategies such as approximation and their use in data analysis and problem solving activities). They also need more opportunities to develop and apply strategies. The small group problem solving described above exemplifies this. Finally, struggling students need intense practice on meaningful skills as well as distributive practice (e.g., systematic instruction in math facts that link to a wider set of number sense activities). Another significant problem with reform based curricular materials is that they do not provide enough practice over time and consequently, some students do not master one concept before another one is introduced.
However, curricular change is not enough. A central theme in *Adding It Up* (Kilpatrick et al., 2001) is that students need to approach mathematics as a sense-making discipline, not one where the primary experience is drill on isolated procedures. Far too often students with LD and those in remedial classrooms spend their time completing worksheets or responding to low level questions in a direct instruction context. Neither of these approaches helps students develop what *Adding It Up* (Kilpatrick et al., 2001) considers a *productive disposition* toward mathematics. These practices do little to help students approach the subject in a conceptually guided fashion or develop the persistence needed to solve complex problems.

What is needed at the level of instructional strategies is similar to many of the practices advocated by mathematics education today, but again, adapted for remedial and special education students. Particularly important is the concept of scaffolding which, as Stone (1998) notes, is overused in special education. Often the term means little more than what process product researchers called direct or explicit feedback decades ago.

A social constructivist view of scaffolding entails much more. Teachers are sensitive to the flow of a mathematical discussion and intervene judiciously. As the problem solving example cited above indicates, they permit students to pursue a range of solutions because complex problems, by their very nature, do not have immediate answers. Scaffolding in this context also entails revoicing important terms and concepts. Whether it is multiplication, the concept of fractions, or a problem solving strategy such as guess and check, teachers can formalize student understanding by rephrasing and refining the meaning of student talk.

Ultimately, scaffolding entails the transfer of responsibility from the teacher to students.

One could argue that at the heart of mathematical proficiency is the development of productive dispositions. Unless remedial and special education students see mathematics as sense-making and useful, it is unlike they will persevere as the topics become increasingly
difficult and more abstract. For this reason alone, the field of special education needs to reconceptualize practices that were perhaps viable 20 years ago but are much less so today.
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Figure 1
Understanding the Number 4

4 as part of visual patterns

4 as one less than 5

4 as the double 2 + 2

4 as a distance from zero on the number line

4 as a quantity or collection of objects
Figure 1 continued

4 as an ordinal number (the fourth face)

4 as a unit of measurement as in \( \frac{1}{4} \) or 4 o’clock

4 = 4  4 = 6 – 2  4 = 12 ÷ 3
4 = 3 + 1  4 = 2 x 2  4 = 3 x 3 - 5

4 as an infinite set of possible outcomes

4 as part of a geometric progression

4 as an arbitrary number

4 + 4 ≠ 8 as in 4 dollars + 4 dimes

different units of 4
Figure 2

Developing Number Sense

Traditional and expanded algorithm and number blocks

\[
\begin{array}{c}
1 \\
47 \\
\hline
+ 24 \\
\hline
71 \\
+ 10 \\
40 \\
70 \\
\hline
74 \\
+ 20 \\
40 \\
\hline
70 \\
+ 7 \\
7 \\
\hline
71 \\
+ 4 \\
4 \\
\hline
71
\end{array}
\]

Horizontal addition by place value

\[
74 + 24 = \\
40 + 7 + 20 + 4 = \\
40 + 20 + 7 + 4 = \\
60 + 11 = \\
60 + 10 + 1 = \\
70 + 1 = 71
\]

“Round up/down strategy”

```
0  10  20  30  40  50
2 4  4 7
47 50
+24
+20
```

“Quarters strategy”

```
0  10  20  25  30  40  47  50
26  47
47 50
+26
+25
```
### Figure 3

The van Hiele Levels for Learning Geometry

<table>
<thead>
<tr>
<th>Level 0:</th>
<th>The student identifies, names, compares and operates on geometric figures (e.g., triangles, angles, intersecting or parallel lines) according to their appearance.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 1:</td>
<td>The student analyzes figures in terms of their components and relationships among components and discovers properties/ rules of a class of shapes empirically (e.g., by folding, measuring, using a grid or diagram).</td>
</tr>
<tr>
<td>Level 2:</td>
<td>The student logically interrelates previously discover properties/ rules by giving or following informal arguments.</td>
</tr>
<tr>
<td>Level 3:</td>
<td>The student proves theorems deductively and establishes interrelationships among networks of theorems.</td>
</tr>
<tr>
<td>Level 4:</td>
<td>The student establishes theorems in different postulational systems and analyzes/ compares these systems.</td>
</tr>
</tbody>
</table>

Fuys, Geddes, & Tischler (1988)
Figure 4

Identifying Different Types of Triangles

A

B

C

D

E

F
Figure 5
The Role of Triangles in Various Two Dimensional Objects

Rectangle

\[
\text{Area} = \text{Base} \times \text{Height} \quad \text{(or 10 x 4 = 40 square units)}
\]

Triangle

Parallelogram

\[
\text{Area} = \text{Base} \times \text{Height}
\]
Figure 5 continued

**Trapezoid**

\[
\text{Area} = \frac{1}{2} \text{Base 1} \cdot \text{Height} + \frac{1}{2} \text{Base 2} \cdot \text{Height}
\]

\[
\text{Area} = \frac{1}{2} (\text{Base 1} + \text{Base 2}) \cdot \text{Height}
\]

**Other Polygons**

**Irregular Pentagon**

Area = sum of areas of 3 triangles

**Regular Hexagon**

Area = 6 \cdot (\frac{1}{2} \cdot \text{base} \cdot \text{height})
Properties of a Circle and Its Area

π = circumference ÷ diameter or 3.14

Area = \( r \cdot r \cdot \pi \) or \( \pi r^2 \)
Figure 6
Working with Three Dimensional Objects

Examples of Prisms

Examples of Pyramids

Volume of a Triangular Prism

Height = stack of triangle bases

Relationship of the Volume of a Cone to a Cylinder with Same Diameter and Height

Contents of 3 cones = 1 cylinder
Problem: “Sara’s father is 60 years old. Sara is ¼ her father’s age. How many years older is Sara’s father than Sara? At what age was Sara 1/10 her father’s age?”

Alicia: “I know what ¼ is. It’s .25. Let me have a calculator. I can figure it out…One fourth of 60 is 15. So Sara is 15 years old.”

Teacher: “That’s good. Does everyone agree? Did she figure out Sara’s age correctly?…OK, what about the second question? Tony, what is the second question asking for?”

Tony: “They want to know when Sara was 1/10 her father’s age.”

Teacher: “So does anyone have a way we can start to solve this problem? Does anyone have a strategy?”

Micha: “We could guess a number.”

Teacher: “OK, so we could use the strategy Guess and Check. Do you want to start Micha?”

Micha: “Well, what if Sara is 1? That would make her father 10.”

Aaron: “But that can’t be because we just said that Sara was 15.”

Micha: “Yeah, but she could still be 1.”

Aaron: “But not if her father was 45 when she was 15. If she is 1 then that’s 14 years ago. That means you’d take away 14 from 45. That wouldn’t be 10.”

Teacher: “Jason, what do you think?”

Jason: “Yeah, she couldn’t be 1 and her dad’s 10. I think Aaron’s right. If you take 15 from 45, that’s 30, so he couldn’t be 10 and Sara is 1.”

Teacher: “Jason, so do you have a strategy for solving the problem? Let’s use the white board to write our numbers.”

Jason: “I would guess she is 7 years younger than her father?”

Teacher: “OK, that would make Sara 8 and her father 51, I mean 55.”

Angela: “That doesn’t work because 7 times 10 isn’t 52.”

Jason: “Then let’s try 5 years younger?”

Teacher: “OK, that would make Sara 10 and her father 50, I mean 55.”

Bryan: “That doesn’t work either because 10 times 10 is 100, not 55.”
Terrell: “Let me try something. Can I do something [with the white board]?”

Terrell: “So I think Sara’s father was 50 and Sara was 5 because 5 times 10 is 50.”

Teacher: “How did you figure that out? What strategy did you use?”

Terrell: “I saw you writing, so I thought I a list would be easier.”

Teacher: “Does everyone think that Terrell’s answer is right? Is that what the question is asking for? Can you tell me how?”

Bryan: “The question says that they want to know when Sara was 1/10 her father’s age.”

Teacher: “And….”

Bryan: “So, if Sara is 5 and her father is 50, then 5 times 10 is 50.”

Teacher: “Or 5 is 1/10 of 50. 50 times 1/10 is 5.”

[Terrell takes white board and writes the two columns below]

<table>
<thead>
<tr>
<th>Sara</th>
<th>Father</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>60</td>
</tr>
<tr>
<td>14</td>
<td>59</td>
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<tr>
<td>13</td>
<td>58</td>
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<td>12</td>
<td>57</td>
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<td>7</td>
<td>52</td>
</tr>
<tr>
<td>6</td>
<td>51</td>
</tr>
<tr>
<td>5</td>
<td>50</td>
</tr>
</tbody>
</table>

[students agree]