The universal Lagrangian for one particle in a potential

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In a system consisting of a single particle in a potential, the classical action $\int L\,dt$ is the number of phase waves that pass through the moving particle, as the particle moves from its initial to its final position. Thus the Lagrangian can be cast into the form $L=p(v_g-v_p)$, where v_g and v_p are the group and phase velocities and p is the momentum.
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I. INTRODUCTION

Hamilton's principal function (now often called the action),

$$S=\int_{t_1}^{t_2} L(q_i,v_i)dt, \tag{1}$$

plays an important role in both classical and quantum mechanics. In classical mechanics, Hamilton's principle,

$$\delta\int_{t_1}^{t_2} L\,dt=0, \tag{2}$$

leads to the Euler–Lagrange equations of motion. In quantum mechanics, S frequently appears as a phase, for example in Wentzel–Kramers–Brillouin calculations and, more fundamentally, in Feynman's sum over histories. However, it is not so clear just what S represents. For example, in a classical mechanical system, why should S be the time integral of the difference between the kinetic and potential energies? Moreover, the Lagrangians for different systems seem to have nothing in common with one another.

The meaning of the action can be made clearer if we connect the classical particle with its underlying de Broglie phase waves. In all that follows the reader will be asked to visualize the classical particle and its phase waves simultaneously. We shall see that, in a single-particle system, S has a simple physical interpretation. Apart from a multiplicative constant, S is the number of phase waves that pass through the moving particle between times t_1 and t_2. Therefore, all single-particle Lagrangians do share a common form.

In Sec. II, we shall see how this theorem can be proven. Section III will illustrate its utility in several applications. In Sec. IV we shall see how to calculate the equations of motion directly from a new form of the Lagrangian. Finally, in Sec. V, we will see that this treatment can be extended to other kinds of waves.

II. AN ALTERNATIVE FORM FOR THE SINGLE-PARTICLE ACTION

The Lagrangian $L(q_i,v_i)$ is a function of the coordinates q_i and their velocities $v_i=dq_i/dt$. By a reversal of the usual Legendre transformation, L may be calculated from the Hamiltonian H:

$$L=\sum_i p_i v_{gi}-H=\mathbf{p}\cdot\mathbf{v}_g-H, \tag{3}$$

where v_g is the particle velocity and the index i labels its components. p_i is the momentum conjugate to q_i. Because the particle velocity is equal to the group velocity of the de Broglie waves, we may also regard v_g as the group velocity: the subscript g will serve as a reminder. Now, for a particle with a well-defined energy,

$$H=\hbar\omega=p\omega/k=pv_p, \tag{4}$$

where $\hbar=h/2\pi$, h is Planck's constant, ω is the frequency, k is the wave number, and v_p is the phase velocity of the de Broglie waves associated with the particle. Using this substitution for H, we obtain

$$L=\mathbf{p}\cdot\mathbf{v}_g-pv_p. \tag{5}$$

If the momentum and particle velocity are parallel, Eq. (5) simplifies to

$$L=p(v_g-v_p). \tag{6}$$

Thus the action becomes

$$S=\int_{t_1}^{t_2} p(v_g-v_p)dt. \tag{7}$$

Now, $(v_g-v_p)dt$ is the relative displacement of the particle and its phase wave. Moreover, $p=h/\lambda$, where λ is the particle's de Broglie wavelength. So we see that the action is equal to h times the number of phase waves that pass through the particle, as the particle moves between the initial and the final position. The term "action" therefore seems rather apt: the action expresses a sort of interaction between the particle and its phase wave.

Most treatments of the relation of the classical action to the quantum mechanical phase wave depend on the rather daunting apparatus of the Hamilton–Jacobi equation,[1] which reflects the route followed historically by Schrödinger. Equation (6) certainly does not replace these approaches, but it does provide another perspective. The action of Eq. (1) is a Lorentz invariant—a fact that usually is established by considerations from relativity. But the Lorentz invariance of the action becomes obvious when we see that the action is made up of countable entities or events. A certain number of phase waves pass through the particle as the particle moves from its initial configuation to its final one. The number of these events cannot depend upon the frame of reference of the observer. Finally, let us note that Eq. (5), or Eq. (6) if the potential is isotropic, provides a universal form of the single-particle Lagrangian.

III. EXAMPLES

A. Nonrelativistic particle in a scalar potential

As our first example, consider a nonrelativistic particle with mass m, momentum $\mathbf{p}$, and potential energy $U(\mathbf{x})$ that varies with position $\mathbf{x}$. The Lagrangian is

$$L=\frac{p^2}{2m}-U=p\left(\frac{p}{2m}-\frac{U}{p}\right). \tag{8}$$

The group velocity is $v_g=p/m$, and the phase velocity is

$$v_p=\frac{\omega}{k}=\frac{H}{p}=\frac{p}{2m}+\frac{U}{p}. \tag{9}$$

Thus the expression in parentheses in Eq. (8) is equal to (v_g-v_p) and the Lagrangian assumes the form of Eq. (6).[2]

B. Relativistic particle in a scalar potential

As a second example of the utility of Eq. (6) for the single-particle Lagrangian, let us see how it can be used to deduce the form of the relativistic Lagrangian for a particle in a scalar potential. In many textbooks, this Lagrangian is often just written down with the remark that it can be verified by showing that its Euler–Lagrange equations are the correct equations of motion.[3] We begin from the expressions for the relativistic momentum p and energy H:

$$p=mv_g\gamma, \tag{10}$$

$$H=mc^2\gamma+U, \tag{11}$$

where $\gamma=(1-v_g^2/c^2)^{-1/2}$ and c is the vacuum speed of light. Then, because $v_p=H/p$, Eq. (6) yields

$$L=-mc^2(1-v_g^2/c^2)^{1/2}-U. \tag{12}$$

This derivation may be the simplest possible way to justify the form of the relativistic Lagrangian.

C. Relativistic particle in an electromagnetic field

If the vector potential $\mathbf{A}$ is introduced, the canonical momentum $\mathbf{p}$ need not be parallel to the group (and particle) velocity $\mathbf{v}_g$. In this case we must use the Lagrangian in the form of Eq. (5). Nevertheless, the action $\int L\,dt$ continues to be the number of phase waves that pass through the moving particle.

To demonstrate this, we first recall that Eq. (5) gives the correct Lagrangian. Let a particle of mass m and charge e move in an electromagnetic field characterized by the vector potential $\mathbf{A}$ and scalar potential ϕ. The canonical momentum $\mathbf{p}$ and energy H are

$$\mathbf{p}=m\mathbf{v}_g\gamma+\frac{e}{c}\mathbf{A}, \tag{13}$$

$$H=mc^2\gamma+e\phi. \tag{14}$$

If we substitute Eqs. (13) and (14) into Eq. (3), we have

$$L=\left(m\mathbf{v}_g\gamma+\frac{e}{c}\mathbf{A}\right)\cdot\mathbf{v}_g-(mc^2\gamma+e\phi) \tag{15}$$

$$=-mc^2\sqrt{1-v_g^2/c^2}-e\phi+\frac{e}{c}\mathbf{v}_g\cdot\mathbf{A}, \tag{16}$$

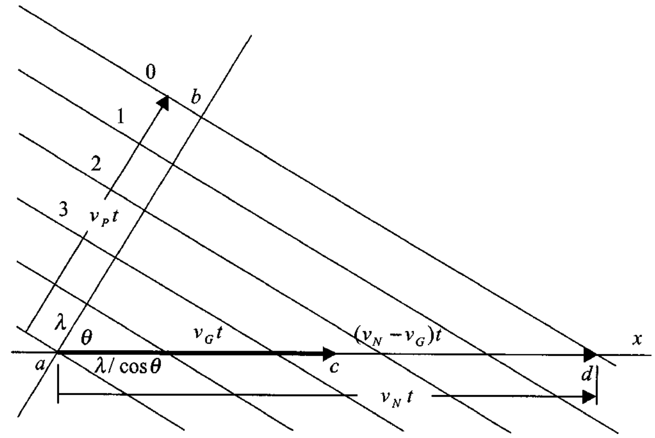

Fig. 1. Phase waves and group velocity in the presence of a vector potential. The direction ab of the wave vector need not coincide with the direction ac of the particle velocity.

the usual form of the Lagrangian for this system. There is nothing new here: H and L are related by the usual prescription.

To complete the demonstration, we must show that, even if the momentum is not in the direction of the group velocity, the action S still represents the number of phase waves that pass through the particle as the particle moves between its initial and final positions. At the outset we recall that the phase velocity v_p is not a vector.[4] However, for the purpose of visualization, v_p may be regarded as directed along $\mathbf{k}$, that is, in the direction of $\mathbf{p}$. So the Lagrangian could also be written as

$$L=\mathbf{p}\cdot(\mathbf{v}_g-v_p\hat{\mathbf{k}}), \tag{17}$$

where $\hat{\mathbf{k}}$ is a unit vector in the direction of $\mathbf{k}$.

Let θ denote the angle between $\mathbf{p}$ and $\mathbf{v}_g$. In Fig. 1, the x axis represents the direction of the particle (and group) velocity $\mathbf{v}_g$, while ab is the direction of $\mathbf{k}$ or $\mathbf{p}$.[5] The wave fronts of the phase waves are at right angles to $\mathbf{k}$. Let phase crest 0 pass through the particle at point a at time 0. In the short time interval t, the phase crest advances a distance v_pt and occupies position bd. Crest 0 now crosses the x axis (that is, the particle trajectory) at point d. Thus the node of intersection between the particle trajectory and the phase wave has moved a distance v_Nt. (This is one reason why v_p is not a vector: its x ''component'' is longer than the vector itself.) From Fig. 1 we see that

$$v_N=v_p/\cos\theta. \tag{18}$$

The distance between two successive crests of the phase wave, measured along the particle trajectory, is $\lambda/\cos\theta$. Now, in time t the particle moves a distance v_gt, advancing from a to c. The number of phase crests that pass through the particle between time 0 and time t is therefore

$$\frac{cd}{\lambda/\cos\theta}=\frac{(v_N-v_g)t}{\lambda/\cos\theta}=\frac{(v_p-v_g\cos\theta)t}{\lambda} \tag{19a}$$

$$=\frac{p}{h}(v_p-v_g\cos\theta)t=(pv_p-\mathbf{p}\cdot\mathbf{v}_g)t/h. \tag{19b}$$

Apart from the factor of h and the overall minus sign involved in the definition of the classical action, the right-hand side of Eq. (19b) is precisely the infinitesimal version of Eq. (1). So, even with the vector potential included, the action is equal to h times the number of phase waves that pass through the particle.

D. Relativistic particle in a strong gravitational field

We have seen that for a particle moving at relativistic speeds in either a scalar or a vector potential, the classical single-particle action is rigorously given by the number of phase waves that pass through the moving particle. This theorem continues to hold even in the curved space of general relativity. As an example, consider a static, isotropic gravitational field represented by the metric

$$ds^2=\Omega^2(\mathbf{x})(c^2dt^2-n^2(\mathbf{x})|d\mathbf{x}|^2), \tag{20}$$

where Ω and n are functions of the spatial coordinates $\mathbf{x}=(r,\theta,\phi)$ or (x,y,z). Many metrics of physical interest can be put into this form, including the Schwarzschild metric. The orbits of massive particles are obtained by requiring that they be geodesics:

$$\delta\int ds=0. \tag{21}$$

If we insert the assumed form of the line element ds, we can express the geodesic condition in the form of Hamilton's principle, Eq. (2), where the Lagrangian is

$$L(x_i,v_{gi})=-mc^2\Omega[1-v_g^2n^2/c^2]^{1/2}. \tag{22}$$

(The factor of the rest mass m has been included for dimensional convenience.)

The canonical momenta are

$$p_i=m\Omega n^2[1-v_g^2n^2/c^2]^{-1/2}v_{gi}. \tag{23}$$

The Hamiltonian is

$$H=mc^2\Omega[1-v_g^2n^2/c^2]^{-1/2}, \tag{24}$$

or, if expressed in terms of the momenta,

$$H=mc^2[\Omega^2+p^2/n^2m^2c^2]^{1/2}. \tag{25}$$

If we square both sides of Eq. (25) and substitute $H=\hbar\omega$ and $p=\hbar k$, we obtain[6]

$$\hbar^2\omega^2=m^2c^4\Omega^2+c^2\hbar^2k^2/n^2. \tag{26}$$

By differentiating both sides of Eq. (26) with respect to k, we obtain the dispersion relation

$$v_pv_g=c^2/n^2, \tag{27}$$

where $v_p=\omega/k$ is the phase velocity and $v_g=d\omega/dk$ is the group velocity of the de Broglie waves.

Now we have all the necessary pieces in hand. Using Eq. (23), we may write Eq. (22) in the form

$$L=p\left[v_g-\frac{c^2}{n^2v_g}\right]. \tag{28}$$

Then, with the use of Eq. (27), L may be put into the form given by Eq. (6). In this example, we have begun with a Lagrangian that is valid for relativistic speeds and arbitrarily strong gravitational fields. Thus, in spite of its simple form, for the class of metrics under consideration, Eq. (6) is an exact general-relativistic relation. Incidentally, Eq. (6) also gives insight into why the geodesics of light rays are null: for light in a gravitational field, the phase and group velocites are equal, so the action vanishes.

IV. CLASSICAL EQUATIONS OF MOTION

It might be thought that having the Lagrangian in the form of Eq. (5) or (6) would be of little utility for calculating equations of motion, because we would need to re-express the momenta p_i in terms of the coordinates q_i and velocities v_{gi}. However, this is not the case. For we can operate directly with the Lagrangian in this form if we regard the p_i as coordinates, along with the q_i. The generalized velocities are then the $\dot{p}_i$ and the v_{gi} (where, of course, $v_{gi}=\dot{q}_i$).[7] We shall regard the phase velocity v_p as a function of the q_i and p_i. (But we could equally well consider v_p to be a function of q_i and v_{gi}.) Then for the single-particle problem we have six Euler–Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v_{gi}}\right)=\frac{\partial L}{\partial q_i}, \tag{29}$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}_i}\right)=\frac{\partial L}{\partial p_i}, \tag{30}$$

where $i=1,2,3$.

Let us consider the q-equation (29) first, and apply it to a Cartesian coordinate, x_i. If we operate on the Lagrangian of Eq. (5) we find:

$$\frac{\partial L}{\partial v_{gi}}=p_i, \tag{31}$$

$$\frac{\partial L}{\partial x_i}=-p\frac{\partial v_p}{\partial x_i}. \tag{32}$$

Thus, the Euler–Lagrange q equation becomes

$$\frac{dp_i}{dt}=-p\frac{\partial v_p}{\partial x_i}. \tag{33}$$

Let us now consider the p equation, (30). Because the Lagrangian does not actually depend upon the generalized velocity $\dot{p}_i$,

$$\frac{\partial L}{\partial \dot{p}_i}=0. \tag{34}$$

Now,

$$\frac{\partial L}{\partial p_i}=v_{gi}-\frac{\partial(pv_p)}{\partial p_i}. \tag{35}$$

Then, because $\partial p/\partial p_i=p_i/p$, Eq. (30) becomes

$$v_{gi}=p\frac{\partial v_p}{\partial p_i}+\frac{p_i}{p}v_p. \tag{36}$$

Equations (33) and (36) are equivalent to Hamilton's canonical equations of motion. Equation (33) corresponds to Hamilton's dynamical equation $dp_i/dt=-\partial H/\partial x_i$, and Eq. (36) corresponds to Hamilton's kinematic equation $\dot{x}_i=\partial H/\partial p_i$. (This correspondence may be easily verified for the particular example of Sec. III A.)

Finally, it will be helpful to express the Lagrangian in an alternative form. If we multiply Eq. (36) by p_i and sum over i, we obtain

$$p\sum_i p_i \frac{\partial v_p}{\partial p_i} = \sum_i p_i v_{gi} - \frac{v_p}{p}\sum_i p_i^2. \tag{37}$$

But because $\Sigma p_i^2 = p^2$, the right-side of Eq. (37) is the Lagrangian itself. Thus

$$L = p\sum_i p_i \frac{\partial v_p}{\partial p_i}. \tag{38}$$

V. GENERALIZATION TO OTHER WAVES

Until now, we have been exclusively concerned with particle mechanics. In the application of our theorem, we have simultaneously pictured a classical particle and its underlying quantum-mechanical phase waves. But the expression of the action as the number of phase waves that pass through the moving particle as it moves from its initial to its final position is broader than mechanics. It applies in the geometrical optics limit to any linear wave system. That is, if it is possible to follow the motion of the center of a wave group, then the equations of motion of the group center can be deduced from a Lagrangian that takes the form of Eq. (5). This may be demonstrated very simply.

Consider a general wave disturbance. The Cartesian components of the group velocity are

$$v_{gi} = \frac{\partial \omega}{\partial k_i}. \tag{39}$$

Because $\omega = k v_p$, we have

$$v_{gi} = k\frac{\partial v_p}{\partial k_i} + v_p \frac{k_i}{k}, \tag{40}$$

which has the form of Eq. (36). If we multiply by k_i and sum over i, we obtain

$$\sum_i k_i v_{gi} = k\sum_i k_i \frac{\partial v_p}{\partial k_i} + v_p k. \tag{41}$$

Since, by analogy to Eq. (38), $k\Sigma k_i \partial v_p / \partial k_i$ may be interpreted as the Lagrangian, we find

$$L = \mathbf{k}\cdot\mathbf{v}_g - k v_p, \tag{42}$$

in complete conformity with Eq. (5). In operating on this Lagrangian to obtain the equations of motion, one should regard v_p as a function of the x_i and k_i.[8] Again, L itself is a function of the x_i, $\dot{x}_i$ $(=v_{gi})$, k_i, and $\dot{k}_i$, although of course the latter do not actually appear.

We have deduced the form of the Lagrangian from the kinematic equation. It is, of course, necessary to show that the Lagrangian also leads to the correct dynamical equation. The q equation for the Lagrangian of Eq. (42) is

$$\frac{d\mathbf{k}}{dt} = -k\nabla v_p. \tag{43}$$

A simple way to see that Eq. (43) is the correct dynamical equation is to examine the familiar special case of light waves in the geometrical optics limit. We take the phase velocity to be an isotropic function of the coordinates alone, so that we may write

$$v_p = c/n, \tag{44}$$

where $n(\mathbf{x})$ is the index of refraction, assumed to be independent of t, so that, for a given frequency component, ω remains constant along the ray. Because $\mathbf{k}$ points along the ray, we may write

$$\mathbf{k} = k\frac{d\mathbf{x}}{ds}, \tag{45}$$

where $d\mathbf{x}$ is a directed element of the ray, and ds is the length of this element of distance taken along the ray. Thus $d\mathbf{x}/ds$ is a unit vector tangent to the ray. Finally, we note that

$$k = \omega n/c. \tag{46}$$

If we substitute Eqs. (44)–(46) into Eq. (43), and treat ω as a constant along the ray, we find

$$\frac{d}{ds}\left(n\frac{d\mathbf{x}}{ds}\right) = \nabla n, \tag{47}$$

which is the standard equation for the shape of a ray in a medium of variable index of refraction.

In this example, in which n does not depend upon the k_i, the wave group follows the rays, that is, travels in the direction of $\mathbf{k}$. This may be seen from Eq. (40) by putting $\partial v_p/\partial k_i = 0$. In the more general case in which v_p depends upon the k_i as well as the x_i, we may first solve Eq. (43) for the shapes of the rays and then apply Eq. (40) to determine the path of the wave group.

VI. CONCLUDING REMARKS

For a wide range of situations, the classical action associated with a one-particle system is the number of phase waves that pass through the particle as the particle moves from its initial to its final position. The applications discussed did not specifically address the question of whether this interpretation continues to hold even in time-dependent potentials. But, in fact, it does. This we might surmise from Sec. III C, in which we did not have to assume that the vector potential A was constant in time. As long as we are able to identify the quantum-mechanical phase wave with the surface of constant action in classical Hamilton–Jacobi theory, the phase speed is rigorously equal to H/p, whether or not the Hamiltonian depends explicitly upon time,[9] and this is the key fact that underlies the theorem. Moreover, the theorem holds also for nonisotropic potentials. Section III D provides a convincing proof. There we chose to work with a particle in a general-relativistic gravitational field by casting the metric into isotropic form. However, the validity of the theorem cannot depend upon this convenience for the following reason. Events are independent of conventions. Thus, if a certain number of phase waves pass through the moving particle as the particle moves from one space–time position to another, this number must be independent of whether or not we choose to work in isotropic coordinates.[10]

The Lagrangians of Eqs. (8), (12), (16), and (22) little resemble one another, yet they do all have something in common, the universal form of Eq. (5). The interpretation of the single-particle Lagrangian as $p(v_g - v_p)$ can often be useful when one needs to write down, or check, a Lagrangian. Moreover, it provides a ready way to visualize the action. Most readers have spent some time at the waterfront watching boat wakes. The next time you get a chance to do this, count the crests of the phase waves as they pass through the wave group. The number that you get is the action.

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[1]Good treatments are found in Edwin C. Kemble, *The Fundamental Principles of Quantum Mechanics* (McGraw–Hill, New York, 1937), pp. 35–51; Wolfgang Yourgrau and Stanley Mandelstam, *Variational Principles in Dynamics and Quantum Theory* (Dover, New York, 1968), pp. 45–64, 116–126.

[2]As with all treatments of phase waves for the nonrelativistic Lagrangian, this is somewhat fortuitous, because the true frequency of the de Broglie waves differs from $H/\hbar$ by the enormous contribution $mc^2/\hbar$ due to the rest mass. However, when the rest mass energy is included, the result is only an effective constant increment of mc^2 to the potential energy U, with no consequences for the validity of the theorem.

[3]For example, Jerry B. Marion and Stephen T. Thornton, *Classical Dynamics of Particles and Systems* (Harcourt Brace Jovanovich, San Diego, 1988), 3rd ed., p. 539; Herbert Goldstein, *Classical Mechanics* (Addison–Wesley, Reading, MA, 1980), 2nd ed., p. 321. In contrast, a very clear and complete derivation is given by Wolfgang Rindler, *Introduction to Special Relativity* (Clarendon, Oxford, 1991), pp. 93–96.

[4]Max Born and Emil Wolf, *Principles of Optics* (Pergamon, Oxford, 1980), 6th ed., p. 18.

[5]The fact that **k** points in the direction of **p**, even when a vector potential **A** is present, is, of course, due to Louis de Broglie. See *Ondes et Mouvements* (Gauthier-Villars, Paris, 1926), pp. 32–35.

[6]For a more detailed justification, see J. Evans, P. M. Alsing, S. Giorgetti, and K. K. Nandi, ‘‘Matter waves in a gravitational field: An index of refraction for massive particles in general relativity,’’ Am. J. Phys. **69**, 1103–1110 (2001).

[7]We are, of course, free to choose the coordinates as we please. Choosing the p_i as coordinates is one standard way, among several, of passing from Euler–Lagrange to Hamiltonian equations of motion. For a detailed discussion, including the fact that this procedure has no consequences for the underlying variational principle, see Goldstein (Ref. 3), pp. 362–365.

[8]This is usually a convenient way to regard v_p. For example, for gravity surface waves on water, the phase velocity depends upon the wave number k and the depth d of the water, $v_p=[(g/k)\tanh(kd)]^{1/2}$. See for example, William C. Elmore and Mark A. Heald, *Physics of Waves* (McGraw–Hill, New York, 1969), p. 187.

[9]For an especially clear and simple proof, see T. T. Taylor, *Mechanics: Classical and Quantum* (Pergamon, Oxford, 1976), pp. 78–82.

[10]An explicit calculation, showing that the theorem holds in an arbitrary metric, has been made by Paul M. Alsing (personal communication).