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# The Optical-Mechanical Analogy in General Relativity: Exact Newtonian Forms for the Equations of Motion of Particles and Photons

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In many metrics of physical interest, the gravitational field can be represented as an optical medium with an effective index of refraction. We show that, in such a metric, the orbits of both massive and massless particles are governed by a variational principle which involves the index of refraction and which assumes the form of Fermat's principle or of Maupertuis's principle. From this variational principle we derive exact equations of motion of Newtonian form which govern both massless and massive particles. These equations of motion are applied to some problems of physical interest.

## 1. INTRODUCTION

The representation of the gravitational field as an optical medium is an old idea, which was exploited by Eddington (Ref. 1, p.109) and which has been developed in more detail by others [2,3]. In many metrics of physical interest, one may find a coordinate transformation that renders the space part of the line element isotropic. If, in addition, the metric has no explicit

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time dependence, the line element may be written in the form

$$\begin{aligned} ds^2 &= \Omega^2(\mathbf{r})c_0^2\,dt^2 - \Phi^{-2}(\mathbf{r})\,[dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\,d\phi^2] \\ &= \Omega^2(\mathbf{r})c_0^2\,dt^2 - \Phi^{-2}(\mathbf{r})\,|d\mathbf{r}|^2, \end{aligned} \tag{1}$$

where $\Omega$ and $\Phi$ are functions of the so-called isotropic coordinates $r$, $\theta$, and $\phi$ (which are related to the standard coordinates by a transformation) and $c_0$ is the vacuum speed of light. Boldface $\mathbf{r}$ is an abbreviation for $(r, \theta, \phi)$. The isotropic coordinate speed of light $c(\mathbf{r})$ at any point in the field may be obtained by putting $ds = 0$:

$$c(\mathbf{r}) = |d\mathbf{r}/dt| = c_0\Phi(\mathbf{r})\Omega(\mathbf{r}). \tag{2}$$

Thus the effective index of refraction is

$$n = \Phi^{-1}(\mathbf{r})\Omega^{-1}(\mathbf{r}). \tag{3}$$

Light trajectories in the gravitational field can be calculated by using the effective index of refraction (3) in any formulation of geometrical optics that happens to be convenient. For example, Wu and Xu have recently shown that the standard differential equation of the ray in classical geometrical optics can be applied to the null geodesic problem [4].

An especially convenient version of geometrical optics is the so-called "$F = ma$" formulation [5,6] in which the equation governing the optical ray assumes the form of Newton's law of motion (acceleration = − gradient of potential energy):

$$d^2\mathbf{r}/dA^2 = \nabla(n^2c_0^2/2). \tag{4}$$

$\mathbf{r}$ is the position of a light pulse moving along the ray. The independent variable (analogous to the time) is the stepping parameter, or optical action $A$. The effective potential energy function is $-n^2c_0^2/2$. All the usual force and energy methods of elementary mechanics can be brought to bear on geometrical optics.

The effective index of refraction (3) (for the Schwarzschild metric, for example) can be used in (4) without modification [7]. In solving problems, one goes into the isotropic coordinates, applies the $F = ma$ optics, then transforms back to the standard coordinates, if desired. The goal of the present paper is to extend these methods to the motion of massive particles.

Fermat's principle has been the subject of renewed interest in general relativity [8–11]. The present paper differs from other recent work in this area in that (i) it focuses on the analogy between the principles of Maupertuis and Fermat in the context of general relativity and (ii) it leads to a remarkable simplification of the equations of motion for both particles and photons.

In Section 2, we derive from the geodesic condition a variational principle which takes on the form of Fermat's principle or Maupertuis' principle. The variational principle, which governs the trajectories of both massive and massless particles, implies that, in isotropic metrics, the particle equations of motion can be cast into the form of Newtonian mechanics or of classical geometrical optics. In Section 3, we derive equations of motion from the variational principle. These equations of motion, which represent a generalization of (4), are exact, apply equally to massive and massless particles, but are nevertheless of Newtonian form. In Section 4, we present effective indices of refraction for a number of metrics of physical and cosmological significance. In Section 5, we illustrate the use of the new equations of motion in some concrete applications: we demonstrate an analogy between two cosmological models and the Maxwell fish-eye lens, we extend some recent calculations involving tests of general relativity in Reissner–Nordström-type metrics, and we present novel derivations of the gravitational and cosmological redshifts.

## 2. TRANSFORMATION OF THE GEODESIC CONDITION

Our goal is to apply the classical optical-mechanical analogy to particle orbits in general relativity. In order to set up the analogy, it will be convenient to begin from a variational principle for the trajectories that can be considered analogous to the principle of Fermat (classical geometrical optics) and the principle of Maupertuis (Newtonian mechanics in velocity-independent potentials).

We shall obtain the variational principle by transformation of the geodesic condition for the particle trajectories,

$$\delta \int_{\mathbf{x}_1,t_1}^{\mathbf{x}_2,t_2} ds = 0, \tag{5}$$

where $\delta$ indicates a variation in the path of integration between two fixed points in spacetime, $(\mathbf{x}_1, t_1)$ and $(\mathbf{x}_2, t_2)$. If we assume the line element can be written in the form (1) this becomes

$$\delta \int_{\mathbf{x}_1,t_1}^{\mathbf{x}_2,t_2} \Omega c_0 [1 - v^2 n^2 / c_0^2]^{1/2} \, dt = 0\,. \tag{6}$$

This is analogous to Hamilton's principle and the effective Lagrangian is

$$L(x_i, \dot{x}_i) = -c_0^2 \Omega [1 - v^2 n^2 / c_0^2]^{1/2}, \tag{7}$$

where $\Omega$ and $n$ are functions of the coordinates alone, where $\dot{x}_i \equiv dx_i/dt$, and where $v^2 = \sum_{i=1}^{3} (dx_i/dt)^2$, if we choose to work in Cartesian coordinates. The expression for the Lagrangian has been multiplied by an extra factor of $-c_0$ for later convenience. (Note: We will always write $c_0$ explicitly. This paper is concerned with an analogy linking geodesic motion, classical geometrical optics, and classical Newtonian mechanics. $c_0$ is not usually suppressed in the latter two fields. Thus the clarity of the analogy is enhanced by retaining classical units of measure.)

The canonical momenta $p_i$ are

$$\begin{aligned} p_i &= \frac{\partial L}{\partial \dot{x}_i} \\ &= \Omega n^2 [1 - v^2 n^2 / c_0^2]^{-1/2} \, \dot{x}_i \, . \end{aligned} \tag{8}$$

The Hamiltonian $H$ may be formed in the usual way,

$$\begin{aligned} H &= \sum_{i=1}^{3} p_i \dot{x}_i - L \\ &= c_0^2 \Omega [1 - v^2 n^2 / c_0^2]^{-1/2}. \end{aligned} \tag{9}$$

Because $\partial L / \partial t = 0$, $H$ is a constant of the motion. If we express $H$ in terms of the $p_i$ rather than the $x_i$ we obtain

$$H = c_0^2 [\Omega^2 + p^2 / n^2 c_0^2]^{1/2}, \tag{10}$$

where $p = |\vec{p}|$. From Hamilton's principle,

$$\delta \int_{\mathbf{x}_1, t_1}^{\mathbf{x}_2, t_2} L \, dt = 0, \tag{11}$$

one may derive in the usual way the corresponding action principle (Jacobi's form of Maupertuis' principle) (Ref. 12, p.125-8,132-4),

$$\delta \int_{\mathbf{x}_1}^{\mathbf{x}_2} \left( \sum_{i=1}^{3} p_i \dot{x}_i \right) \, dt = 0, \tag{12}$$

where now the path of integration is varied between two fixed points in space, $\mathbf{x}_1$ and $\mathbf{x}_2$, where the energy must be held constant on the varied paths, but where the times at the end points need not be held fixed. With the canonical momenta from (8), this becomes

$$\delta \int_{\mathbf{x}_1}^{\mathbf{x}_2} n^2 v^2 \Omega [1 - v^2 n^2 / c_0^2]^{-1/2} \, dt = 0 . \tag{13}$$

We restrict the varied paths to those that satisfy the energy constraint by substituting the constant H for the right side of (9) where this appears in (13). Then, putting $dt = dl/v$, where $dl = |d\mathbf{r}| = (\sum_{i=1}^{3} dx_i^2)^{1/2}$ we obtain

$$\delta \int_{\mathbf{x}_1}^{\mathbf{x}_2} n^2 v \, dl = 0 . \tag{14}$$

This is a variational principle on which an analogy to geometrical optics or to classical mechanics can be constructed. In obtaining (14) we have preferred, for the sake of directness, clarity and and consistency of notation, to begin from the fundamental principle (5). But (14) may also be derived from other versions of the three-dimensional variational principle for particle orbits in static metrics, for example, the forms first obtained by Weyl [13] and Levi-Civita [14].

In (14), $n^2 v$ is to be considered a function of position alone. The path of integration is varied between the fixed end points $\mathbf{x}_1$ and $\mathbf{x}_2$, and the value of $H$ is held constant during the variation. Thus, (14) is of the same form as Fermat's principle, which forms a basis for classical geometrical optics, and Maupertuis' principle, which forms a basis for classical mechanics (as long as the force can be derived from a velocity-independent potential):

| Relativistic gravitational mechanics | Geometrical optics (Fermat) | Classical mechanics (Maupertuis) |
|---|---|---|
| $\delta \int n^2 v \, dl = 0$ | $\delta \int n \, dl = 0$ | $\delta \int v \, dl = 0 .$ |

In the context of motion in a static gravitational field, both Fermat's principle and Maupertuis' principle are simply special cases of (14). For the null geodesics, i.e., for the paths of light, the derivation given above must be slightly modified, to keep each step well defined. But the final result is too well known to require detailed discussion here: in static metrics, light obeys Fermat's principle. That is, the path taken by light between two fixed points in space is one for which the coordinate time of travel is stationary (Ref. 15, Ref. 16, p.99-100). In the language of a refractive index, this may be written $\delta\int n\,dl = 0$. Since, for light, $v = c_0/n$, (14) does reduce to the appropriate form. To obtain Maupertuis' principle (and hence Newtonian gravitational motion), note that in ordinary solar-system dynamics, we may put $n^2 \approx 1$. That is, in the Newtonian limit, $n^2$ may be treated as constant in the variational calculation and we obtain Maupertuis' principle as the classical limit of (14).

## 3. EXACT EQUATIONS OF MOTION OF NEWTONIAN FORM

Let the path of the particle be parametrized by a stepping parameter $A$. That is, at each point on the path, the three space coordinates $\mathbf{r}$ (and also the time $t$) are regarded as functions of $A$. We defer for the moment choosing $A$: we shall define $A$ to get the simplest equations of motion. Thus we write (14) in the form

$$\delta\int_{\mathbf{x}_1}^{\mathbf{x}_2} n^2 v\left|\frac{d\mathbf{r}}{dA}\right| dA = 0,$$

where $|d\mathbf{r}/dA| = [\sum_{i=1}^{3}(dx_i/dA)^2]^{1/2}$.

Let $\mathbf{r}(A)$ denote the true path. To obtain a varied path, we replace $\mathbf{r}(A)$ by $\mathbf{r}(A) + \mathbf{w}(A)$, where $\mathbf{w}(A)$ is an arbitrary, infinitesimal vector function, subject to the constraint that $\mathbf{w} = 0$ when $A$ is such that $\mathbf{r} = \mathbf{x}_1$ or $\mathbf{x}_2$. That is, the variation must vanish at the end points. Now

$$\delta\int n^2 v\left|\frac{d\mathbf{r}}{dA}\right| dA = \int[\delta(n^2 v)]\left|\frac{d\mathbf{r}}{dA}\right| dA + \int(n^2 v)\left(\delta\left|\frac{d\mathbf{r}}{dA}\right|\right)dA + \int n^2 v\left|\frac{d\mathbf{r}}{dA}\right| \delta dA. \tag{16}$$

Calculating the two variation in the first term on the right-hand side of (16), we have

$$\delta(n^2 v) = \nabla(n^2 v)\cdot\mathbf{w}. \tag{17}$$

In calculating the variation in the second term of (16) it is important to remember that the change to the varied path will, in general, also produce a change in $A$. Thus

$$\begin{aligned}\delta\left|\frac{d\mathbf{r}}{dA}\right| &= \left|\frac{d\mathbf{r}+d\mathbf{w}}{dA+\delta dA}\right| - \left|\frac{d\mathbf{r}}{dA}\right| \\ &= \frac{d\mathbf{r}}{dA}\cdot\frac{d\mathbf{w}}{dA}\left|\frac{d\mathbf{r}}{dA}\right|^{-1} - \left|\frac{d\mathbf{r}}{dA}\right|\frac{\delta dA}{dA},\end{aligned} \tag{18}$$

to first order in the variation. Substituting (17) and (18) into (16), we find

$$\delta\int n^2 v\left|\frac{d\mathbf{r}}{dA}\right| dA = \int\left[\left|\frac{d\mathbf{r}}{dA}\right|\nabla(n^2v)\cdot\mathbf{w} + n^2v\left|\frac{d\mathbf{r}}{dA}\right|^{-1}\frac{d\mathbf{r}}{dA}\cdot\frac{d\mathbf{w}}{dA}\right] dA.$$

Note that the terms involving $\delta dA$ have cancelled. This was to be expected, since (14) shows that the integral does not actually depend upon the range in $A$ or, indeed, on what we select to use as parameter. Integrating the term involving $d\mathbf{w}/dA$ by parts, and using the fact that $\mathbf{w}$ must vanish at the endpoints, but is otherwise arbitrary, we arrive at the differential equation that must be satisfied by the particle trajectory:

$$\left|\frac{d\mathbf{r}}{dA}\right|\nabla(n^2v) - \frac{d}{dA}\left(n^2v\left|\frac{d\mathbf{r}}{dA}\right|^{-1}\frac{d\mathbf{r}}{dA}\right) = 0\,. \tag{19}$$

This differential equation plays the role of an equation of motion. Another way to obtain (19) is to parametrize the path by one of the Cartesian coordinates (say $z$), rather than $A$, since the variation in $z$ must vanish at $\mathbf{x}_1$ and $\mathbf{x}_2$. In this case, one writes

$$\delta\int n^2v\frac{dl}{dz}\,dz = 0\,.$$

One may then simply write down the Euler conditions for the integral to be stationary, and then transform from $z$ to $A$ as independent variable. The result will be the same, that is (19).

To give the equation of motion the simplest possible form, and to take advantage of the analogy to Newtonian mechanics, let us now define $A$ by

$$\left|\frac{d\mathbf{r}}{dA}\right| \equiv n^2v. \tag{20}$$

With this definition of $A$, the equation of motion (19) becomes

$$\frac{d^2\mathbf{r}}{dA^2} = \nabla\left(\frac{1}{2}\,n^4v^2\right). \tag{21}$$

Equation (21) is the generalization of (4) that was sought. The left-hand side of (21) is of the form of an acceleration: it is the second derivative of the position vector with respect to the independent variable. The right-hand side of the equation is of the form of a force: $-\frac{1}{2}n^4v^2$ plays the role of a "potential energy function." The analogue of the velocity is $d\mathbf{r}/dA$. Thus the analogue of the kinetic energy is $|d\mathbf{r}/dA|^2$. The analogue of the total energy is the sum of the potential and the kinetic. But, by virtue of eq. (20), these two are guaranteed to sum to zero:

$$\tfrac{1}{2}|d\mathbf{r}/dA|^2 - \tfrac{1}{2}n^4v^2 = 0\,. \tag{22}$$

Thus the calculation of the paths of light and of massive particles in general relativity reduces to the zero-energy $F = ma$ optics of [5]. It is to be noted that the "conservation of energy" condition (22) amounts to a restatement of the definition (20) of $A$.

The optical-mechanical analogy, embodied in (21) and (22), provides an exact treatment in Newtonian form of the motion of massive particles, as well as light, in general relativity. The Newtonian form should be thought of as coming from $F = ma$ optics (which is exact) and not from Newtonian mechanics (which is, of course, only approximate). Equations (21) and (22) allow one to handle the paths of light and of the planets as if they existed in a flat three-dimensional space. Other approaches to this goal are, of course, possible [17], but the treatment presented here has three advantages: simplicity, complete conformity to the equations of Newtonian mechanics, and a uniform treatment of both light and massive particles. This treatment has a reasonably high degree of generality and is applicable whenever the line element can be written in the form (1).

In solving problems with (21) or (22), one may use without modification all the familiar methods of Newtonian mechanics. One simply thinks of $A$ as if it were the time. Moreover, rather than beginning from (21) or (22) in every case, one may often simply write down an exact general-relativistic formula by analogy to the corresponding classical formula. For the motion of both light and massive particles in static metrics, one may begin from any classical Newtonian formula describing the motion of particles in static, velocity-independent potentials. The correct general-relativistic expressions will be obtained if one makes the following transcriptions in the classical formulas:

$$t \to A, \qquad U \to -n^4v^2/2, \qquad E \to 0\,. \tag{23}$$

The spatial coordinates $(x, y, z)$, or $(r, \theta, \phi)$, etc., transcribe as themselves. Of course, it must be kept in mind that the formulas written down in this way apply in the isotropic coordinate system. After the equations governing the situation are obtained, or after they are solved, one may transform back to the original metric, if desired. A second point of vital importance is that the analogue of the classical energy $E$ is always the number zero. Thus, the constant of the motion $H$ plays no role in the analogy. $H$ should rather be regarded as a parameter: the potential energy function depends upon $H$ as well as the coordinates. A third point to be stressed is that the exact general-relativistic formulas written down by analogy to the classical formulas will always apply with equal validity to both massive and massless particles.

For both light and particles, the stepping parameter $A$ is defined by (20). In many calculations, e.g., in finding the shape of an orbit, the stepping parameter is ultimately eliminated. Equation (20) will suffice for this purpose. In other situations (for example, in a radar-echo delay calculation), it may be necessary to have an explicit connection between $A$ and $t$. Note that

$$\left|\frac{d\mathbf{r}}{dA}\right| = \left|\frac{d\mathbf{r}}{dt}\right|\frac{dt}{dA} = v\,\frac{dt}{dA}\,.$$

Substituting in (20) gives

$$dA = dt/n^2. \tag{24}$$

Thus the stepping parameter is the same as that used in the $F = ma$ formulation of geometrical optics. $A$ is called the optical action because $dA$ is proportional to $c(\mathbf{r})\,dl$, and thus is analogous to the action $v(\mathbf{r})\,dl$ of classical mechanics.

The only difference between the treatment of light and that of particles resides in the choice of $v(\mathbf{r})$, which forms a part of the effective potential energy $-n^4v^2/2$. For light,

$$v = c_0 n^{-1} \qquad \text{(light)}, \tag{25}$$

But for massive particles, (9) gives

$$v = c_0 n^{-1}[1 - c_0^4\Omega^2/H^2]^{1/2} \qquad \text{(particles)}. \tag{26}$$

In (26), $H$ is a constant parameter determined by the initial conditions, while $n$ and $\Omega$ are functions of the spatial coordinates determined by the metric. Because the particle expression for $v(\mathbf{r})$ contains the parameter $H$, the particle problem has an extra degree of freedom: we may specify the initial speed of the particle. Thus, in general, more types of orbits exist for massive particles than for light in the same metric.

For a particle in empty space devoid of gravitational influences, $\Omega \approx 1$, $n \approx 1$, and (9) becomes

$$H \approx c_0^2(1 - v^2/c_0^2)^{-1/2} = c_0^2\gamma. \tag{27}$$

In the solar-system dynamics of the Schwarzschild metric, $v/c_0 \ll 1$ and [see (39) and (43)] $\Omega \approx 1 - m/r$ so (9) becomes

$$H \approx c_0^2 + \tfrac{1}{2}v^2 - m/r. \tag{28}$$

That is, in classical planetary orbits, $H$ is approximately equal to $c_0^2 + E$, the rest-mass energy plus the classical kinetic and potential energy per unit mass.

The classical optical-mechanical analogy is based upon the similarity of form of the principles of Fermat and Maupertuis. Usually this analogy is expressed in terms of nonlinear partial differential equations. In the form of the analogy originally due to William Rowan Hamilton, the eikonal equation of geometrical optics corresponds to the (time-independent) Hamilton–Jacobi equation of particle mechanics. But in fact the analogy is far more general. Corresponding to every possible formulation of classical mechanics, there is an analogous formulation of geometrical optics (and vice versa). Thus, as in (4), we can cast geometrical optics into the form of Newton's law of motion.

The variational principle (14) has permitted us to extend the analogy to the geodesic problem for both light and particles in isotropic metrics. Our discussion of the optical-mechanical analogy in general relativity has stressed Newtonian forms. But, corresponding to every formulation of *either* classical particle mechanics *or* of classical geometrical optics, there will be an analogous formulation of the geodesic problem in general relativity. Few of these classical models for the reformulation of the geodesic equations of motion lead to any special insight or simplification. The economy of expression and simplicity of form embodied in (21) and (22) depend, not so much on the formulation of mechanics (or of geometrical optics) that is chosen as model, as on the use of $A$ rather than $t$ as independent variable. The remainder of the paper is devoted to the development of calculation techniques based on the Newtonian formulation embodied in (21) and (22).

## 4. INDICES OF REFRACTION FOR SOME IMPORTANT METRICS

### 4.1. Metrics of the Reissner–Nordström (RN) type

A number of line elements of physical interest assume the following form in standard coordinates $(t, r', \theta, \phi)$:

$$ds^2 = c_0^2\left[1 - \frac{2m}{r'} + \frac{\beta}{r'^2}\right]dt^2 - \left[1 - \frac{2m}{r'} + \frac{\beta}{r'^2}\right]^{-1} dr'^2 - r'^2 d\theta^2 - r'^2 \sin^2\theta\phi^2, \qquad (29)$$

where

$$m = MG/c_0^2, \qquad (30)$$

$M$ is the mass of the central gravitating body, $G$ is the gravitation constant, and $\beta$ is another parameter. We wish to write the line element in terms of isotropic coordinates $(t, r, \theta, \phi)$. We will indicate briefly how to effect the transformation, using a systematic technique (Ref. 18, p.174-177). The idea is to express the spatial part as $-\Phi^{-2}(r)[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2]$, where $\Phi(r)$ is yet to be determined. Equating the angular and the radial parts of the two line elements, we have

$$r'^2 = \Phi^{-2} r^2 \qquad (31)$$

and

$$\left[1 - \frac{2m}{r'} + \frac{\beta}{r'^2}\right]^{-1} dr'^2 = \Phi^{-2} dr^2. \qquad (32)$$

If we divide (32) by (31) to eliminate $\Phi$, then integrate and use the condition that at large radial distances $r$ and $r'$ must be asymptotically equal, we obtain

$$2r = (r' - m) + (r'^2 - 2mr' + \beta)^{1/2}. \qquad (33)$$

The inverse transformation is

$$r' = r + m + (m^2 - \beta)/4r. \qquad (34)$$

Using these transformations, the line element (29) can be expressed in the form of (1), with

$$\Omega^2(r) = [1 - (m^2 - \beta)/4r^2]^2[1 + m/r + (m^2 - \beta)/4r^2]^{-2} \qquad (35)$$

$$\Phi^{-2}(r) = [1 + m/r + (m^2 - \beta)/4r^2]^2. \qquad (36)$$

The effective refractive index $n(r)$ is

$$n(r) = [1 + m/r + (m^2 - \beta)/4r^2]^2[1 - (m^2 - \beta)/4r^2]^{-1}. \qquad (37)$$

It is also helpful to have expressions for $\Phi$, $\Omega$ and $n$ in terms of the standard radial coordinate $r'$. Let $u \equiv 1/r$ and $u' \equiv 1/r'$. Then it is easy to show that

$$\Phi^2(u') = \tfrac{1}{4}[1 - mu' + (1 - 2mu' + \beta u'^2)^{1/2}]^2 \qquad (38)$$

$$\Omega^2(u') = 1 - 2mu' + \beta u'^2. \qquad (39)$$

And of course

$$n(u') = \Phi^{-1}(u')\Omega^{-1}(u'). \qquad (40)$$

In transforming coordinates, it is often helpful to use

$$du = n\,du' \qquad \text{or} \qquad dr = \Phi\Omega^{-1}dr', \qquad (41)$$

together with

$$u = \Phi^{-1}u' \qquad \text{or} \qquad r = \Phi r'. \qquad (42)$$

The singularities of $n(r)$, or, equivalently, the horizons of the spacetime, occur at $r_s = (m/2)(1 - \beta/m^2)^{1/2}$, provided that $\beta/m^2 \leq 1$. Therefore, the expression for $n(r)$ is valid in the region $r > r_s$. If $\beta = m^2$, $r_s = 0$; i.e., the event horizon shrinks to zero size. In this case, $n(r) = 1 + m/r$ and is regular everywhere for $r > 0$. If $\beta > m^2$, the function $n(r)$ is not singular anywhere, since $r_s$ becomes imaginary. Let us now examine some special cases of the metric (29).

*Schwarzschild exterior metric*

The Schwarzschild exterior metric applies to the spacetime around an electrically neutral, static, spherical mass $M$. In this case, (29)–(42) apply (Ref. 19, p.840, Ref. 20, p.515-521) with

$$\beta = 0\,. \qquad (43)$$

*Reissner–Nordström* (RN) *metric*

The gravitational field due to an electrically charged, static spherical mass $M$ is given by the RN solution of Einstein's field equations. In this case, (29)–(42) apply with

$$\beta = GQ^2/c_0^4\,, \qquad (44)$$

where $Q$ is the charge on the central body.

*Bertotti–Robinson* (BR) *metric*

The BR metric describes a universe filled with electromagnetic radiation of uniform density and uniformly random direction [21]. In this case, (29)–(42) apply with

$$\beta = m^2 \tag{45}$$

where $m$ is now a nonphysical effective point mass. The BR solution may also be obtained as a special case of the metric obtained recently by Halilsoy.

*Halilsoy metric*

The Halilsoy metric describes spacetime around a static, uncharged, spherically symmetric mass $M$ which is embedded in an externally created electromagnetic field [22,23]. Equations (29)–(42) apply with

$$\beta = q^2 m^2 \tag{46}$$

where $0 \leq q \leq 1$, and where $q$ represents the measure of the external electromagnetic field.

*Soleng metric*

The Soleng metric represents the gravitational field due to a central mass $M$ surrounded by a field having a traceless energy-momentum tensor $T^{\mu}{}_{\nu} = f(r)\,\mathrm{diag}\,[1, 1, -1, -1]$. Recently, such a $T^{\mu}{}_{\nu}$ has been interpreted as the energy-momentum tensor associated with an anisotropic vacuum [24–26]. Here (29)–(42) apply with

$$\beta = 6\,\delta\, m^2 \tag{47}$$

where $\delta$ is the Soleng parameter, which determines the effective energy density of the anisotropic vacuum.

**4.2. de Sitter universe and the Maxwell fish-eye**

The de Sitter line element in standard coordinates is

$$ds^2 = (1 - \Lambda r'^2/3)c_0^2\, dt^2 - (1 - \Lambda r'^2/3)^{-1} dr'^2 - r'^2 d\theta^2 - r'^2 \sin^2\theta\, d\phi^2, \tag{48}$$

where $\Lambda$ is the cosmological constant, which is proportional to the space curvature. $\Lambda$ can be positive or negative, corresponding to a closed or an open de Sitter universe (Ref. 27, p.346-349).

To pass over to isotropic coordinates, we may use the method outlined above, together with the requirement that for small radial distances the new radial variable $r$ should asympotically approach $r'$. The result is the well-known transformation

$$r' = r(1 + \Lambda r^2/12)^{-1}. \tag{49}$$

Then, in the isotropic coordinates,

$$\begin{aligned} ds^2 = {} & (1 - \Lambda r^2/12)^2(1 + \Lambda r^2/12)^{-2} C_0^2 dt^2 \\ & - (1 + \Lambda r^2/12)^{-2}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2). \end{aligned} \tag{50}$$

The effective index of refraction is

$$n(r) = (1 - \Lambda r^2/12)^{-1}. \tag{51}$$

This index of refraction is valid for either positive or negative $\Lambda$, with $r$ defined through (49). Let us examine the case $\Lambda < 0$, corresponding to the open de Sitter universe. Let us write $\Lambda = -K$, where $K$ is then a positive constant. The effective index of refraction of the open de Sitter universe, in the isotropic coordinates, is then

$$n(r) = (1 + K r^2/12)^{-1}. \tag{52}$$

This index of refraction is of exactly the same form as the index encountered in a traditional problem of classical geometrical optics — the Maxwell fish-eye lens. The index of refraction in the Maxwell fish-eye is

$$n_{\mathrm{M}}(r) = n_0(1 + r^2/a^2)^{-1}. \tag{53}$$

in which $a$ and $n_0$ are constants. Comparing (52) and (53), we note that the open version of the de Sitter universe is a Maxwell fish-eye lens with $n_0 = 1$ and $a^2 = 12/K$.

### 4.3. Robertson–Walker universe

The Robertson–Walker (RW) metric represents the gravitational field in a homogeneous and isotropic universe. In the standard comoving coordinates $(t, r', \theta, \phi)$, the RW line element is given by

$$ds^2 = c_0^2 \, dt^2 - R^2(t)\left[\frac{dr'^2}{1 - kr'^2} + r'^2 d\theta^2 + r'^2 \sin^2\theta \, d\phi^2\right], \tag{54}$$

in which $R(t)$ is a dimensionless scale factor and $k$ is a constant with dimensions of (length)$^{-2}$. We may pass over to isotropic coordinates by the usual method and requiring that for small radial distances the new radial coordinate $r$ should asymptotically be equal to $r'$. The result is the well-known transformation

$$r' = r(1 + kr^2/4)^{-1}. \tag{55}$$

In the isotropic coordinates $(t, r, \theta, \phi)$, the line element is

$$ds^2 = c_0^2\, dt^2 - R^2(t)(1 + kr^2/4)^{-2}|d\mathbf{r}|^2. \tag{56}$$

Defining the refractive index $n$ in the usual way, we obtain

$$n = \frac{R(t)}{1 + kr^2/4}\,. \tag{57}$$

For the case $k > 0$, corresponding to a closed RW universe, and for a fixed cosmological epoch $t = t_0$, this corresponds to the index of refraction (53). Thus the closed Robertson–Walker universe is a Maxwell fish-eye lens with $n_0 = R(t_0)$ and $a^2 = 4/k$. We shall see below that the correspondence between the Maxwell fish-eye and the Robertson–Walker universe does not actually demand that we restrict the latter to a particular moment $t_0$.

## 5. SOME APPLICATIONS

### 5.1. Central-force motion

Many metrics of interest — including all those discussed in Section 4 — are spherically symmetric. In such a case, $n, v, \Omega$ and $\Phi$ are functions of the radial coordinate alone. The orbit (whether of light or of a massive particle) lies in a plane containing the force center and there is a constant of the motion analogous to the angular momentum. Let $\phi$ be measured in the plane of the motion. Then, from (21),

$$r^2 d\phi/dA \equiv h = \text{constant}. \tag{58}$$

Note that $h$ is related to the classical-mechanical angular momentum per unit mass $h_0$ ($\equiv r^2 d\phi/dt$) by

$$h = n^2 h_0\,. \tag{59}$$

Now we may easily obtain general-relativistic analogues of the standard formulas of classical central-force motion. In (22), which is the analogue of the classical conservation of energy condition, we may write out

$|dr/dA|^2$ in plane-polar coordinates, then eliminate $A$ by means of (58). The orbit shape $\phi(r)$ is thereby reduced to an integration:

$$\phi = h \int^r r^{-2}[n^4 v^2 - h^2/r^2]^{-1/2}\, dr. \tag{60}$$

The classical limit of (60) is the familiar equation

$$\phi = h_0 \int^r r^{-2}[2(E-U) - h_0^2/r^2]^{-1/2}\, dr.$$

Note that we could have immediately written down (60), which is an exact general-relativistic expression, on the model of the classical expression, simply by using the transcriptions (23), together with $h_0 \to h$ (which follows from $t \to A$). Moreover, (60) applies both to light and to massive particles. To apply (60) to either massless or massive particles, we need only insert the appropriate specific form (25) or (26) for $v(r)$.

Another form of the orbit equation is frequently useful. Let $u = 1/r$. Then, in analogy to the classical formula

$$\frac{d^2u}{d\phi^2} + u = -h_0^{-2}\frac{dU}{du},$$

we must have in general relativity

$$\frac{d^2u}{d\phi^2} + u = h^{-2}\frac{d}{du}(n^4v^2/2), \tag{61}$$

which, again, applies to both particles and photons. We have written down (61) simply by analogy to classical mechanics. But it may also be obtained by beginning with the radial component of (21) and eliminating $A$.

A third useful form of the orbit equation is

$$h^2[(du/d\phi)^2 + u^2] - n^4v^2 = 0. \tag{62}$$

### 5.2. Light rays in the de Sitter and Robertson–Walker universes

As noted above, the open de Sitter universe is equivalent to a traditional problem in classical geometrical optics — Maxwell's fish-eye lens. It follows (i) that the open de Sitter universe constitutes an absolute optical instrument and (ii) that, in the system of isotropic coordinates, the rays are eccentric circles.

Beginning from the orbit equation (60) and the index of refraction (52) and integrating, we obtain the polar equation for the light ray in the open de Sitter universe:

$$\sin(\phi - \alpha) = \frac{h(Kr^2 - 12)}{r(144c_0^2 - 48h^2K)^{1/2}}, \tag{63}$$

where $\alpha$ is a constant of integration. In effecting this calculation, we can follow step-for-step the calculation of ray shapes in the classical Maxwell fish-eye (Ref. 28, p.147-149). Since $(Kr^2 - 12)/r\sin(\phi - \alpha)$ =constant, we can write the equation for a family of light rays passing through a fixed point $P_0(r_0, \phi_0)$ as

$$\frac{Kr^2 - 12}{r\sin(\phi - \alpha)} = \frac{Kr_0^2 - 12}{r_0\sin(\phi_0 - \alpha)}. \tag{64}$$

For any value of $\alpha$, this equation is satisfied at point $P_1 = (r_1, \phi_1)$ where $r_1 = 12/Kr_0$ and $\phi_1 = \phi_0 + \pi$. This shows that all the rays from an arbitrary point $P_0$ meet at a point $P_1$ on the line joining $P_0$ to the origin $O$ such that $OP_0 \cdot OP_1 = 12/K$. Hence the imaging in Maxwell's fish-eye lens is an inversion. From any point $P_0$ in the three-dimensional space an infinity of rays originate which are then focused at an image point $P_1$. The images are therefore *sharp* (stigmatic). (In most real optical instruments, of the infinity of points passing thorugh an object point, only a finite number pass through the image point, the other rays only passing near the image point. Such images are not sharp ones.) Now, an instrument which sharply focuses an image of a three-dimensional region of space is called an absolute optical instrument. Thus, the open de Sitter universe constitutes an absolute optical instrument. All the theorems pertaining to absolute optical instruments apply. For example, the optical length of a line segment in the image must be equal to the optical length of the corresponding line segment in the object (Ref. 28, p.143-147).

Moreover, by analogy to the Maxwell fish-eye, a ray in the de Sitter universe is a circle in the isotropic coordinate system. This goes exactly as in the classical geometrical optics treatment of the Maxwell fish-eye. On the left-hand side of (63), write $\sin(\phi - \alpha) = \sin\phi\cos\alpha - \cos\phi\sin\alpha$, then put $x = r\cos\phi$ and $y = r\sin\phi$. Then (63) may be written in the form

$$(x + b\sin\alpha)^2 + (y - b\cos\alpha)^2 = 12/K + b^2 \tag{65}$$

where $b = (hK)^{-1}(36c_0^2 - 12h^2K)^{1/2}$. Thus each ray is a circle. Note that in (63) if we put $K = 0$, we obtain a straight-line ray, as we would expect:

$$r\sin(\phi - \alpha) = \text{constant}.$$

These results may be extended with a little modification to the closed Robertson–Walker universe. The formalism developed in this paper was designed for static metrics, so it may seem at first sight that we cannot deal with the RW case. However, in the case of the null geodesic, a time-dependent conformal factor in the metric need not affect the basic procedure or the most important conclusions. (The same cannot be said of the particle trajectories.) Let us see how this works out in the language of refractive indices.

As noted above, the closed Robertson–Walker universe yields a factorable index of refraction,

$$n = n_{\mathrm{s}}(\mathbf{r})n_{\mathrm{t}}(t), \tag{66}$$

in which $n_{\mathrm{s}}$ is a function of the spatial coordinates alone and $n_{\mathrm{t}}$ is a function of the time alone. For the RW metric in isotropic comoving coordinates

$$n_{\mathrm{s}} = (1 + kr^2/4)^{-1} \tag{67}$$

$$n_{\mathrm{t}} = R(t). \tag{68}$$

The bending of light rays depends only on the spatial gradient of $n$. The fact that $n$ varies everywhere in space with the same multiplicative function of time $R(t)$ does not affect the shape of a light ray. One way to see this is to consider Snell's law. If the index of refraction factors into the form (66), then whenever Snell's law is applied at an interface between regions of different $n$, the common factor $n_{\mathrm{t}}(t)$ will cancel from the two sides of the equation. More formally, we can define a new time coordinate $\tau$ by $dt = R d\tau$. Then the line element (56) becomes

$$ds^2 = c_0^2 R^2\, d\tau^2 - R^2(1 + kr^2/4)^{-2}|d\mathbf{r}|^2.$$

and the effective index of refraction becomes simply

$$n = n_{\mathrm{s}} = (1 + kr^2/4)^{-1} \tag{69}$$

The scale factor $R(\tau)$ or $R(t)$ thus can have no effect on the shape of a ray in the isotropic, comoving coordinates. $R$ only influences the progress in time of light along the ray.

As far as ray shapes are concerned, then, the RW universe is entirely analogous to the Maxwell fish-eye lens. It follows (i) that the closed Robertson–Walker universe also constitutes an absolute optical instrument and (ii) that, in the system of isotropic, comoving coordinates, the rays are eccentric circles.

### 5.3. Light and particle motion in RN-type metrics

In this section, we shall illustrate the use of the Newtonian forms of the orbit equations in some applications to RN-type metrics. In particular, we calculate the effect of the parameter $\beta$ (29) on three tests of general relativity. Our calculations will supplement and extend those of Halilsoy [23].

We may begin from (62). Inserting (26) for $v(r)$ in the second term, we obtain

$$\left[\frac{du}{d\phi}\right]^2 + u^2 - (c_0/h)^2 n^2 [1 - c_0^4 H^{-2} \Omega^2] = 0\,. \tag{70}$$

This differential equation is exact, but it may not appear very familiar. We may transform back to the original (nonisotropic) coordinates by using (41) and (42) in the first two terms of (70), with the result

$$\left[\frac{du'}{d\phi}\right]^2 + u'^2\Omega^2 - (c_0/h)^2 [1 - c_0^4 H^{-2} \Omega^2] = 0\,. \tag{71}$$

Substituting (39) for $\Omega^2(u')$, then differentiating with respect to $\phi$, we obtain

$$\frac{d^2u'}{d\phi^2} + u' - \frac{mc_0^6}{h^2H^2} = -\frac{\beta c_0^6}{h^2H^2}\, u' + 3mu'^2 - 2\beta u'^3. \tag{72}$$

*Bending of light rays*

The equation for the shape of a light ray results from letting $H \to \infty$ in (72):

$$\frac{d^2u'}{d\phi^2} + u' = 3mu'^2 - 2\beta u'^3. \tag{73}$$

This equation may be solved by the usual perturbative method. If the right side of (73) is temporarily put equal to zero, we obtain the straight-line solution

$$u' = \frac{\sin\phi}{R},$$

where $R$ is the distance of closest approach to the origin. Substituting the zeroth-order solution $\sin\phi/R$ for $u'$ on the right side of (73) and solving the resulting differential equation for $u'(\phi)$, we obtain the solution of first order in $m$ and $\beta$:

$$u' = \frac{\sin\phi}{R} + \frac{3m}{2R^2}\left[1 + \frac{1}{3}\cos 2\phi\right] + \frac{3\beta}{4R^3}\,\phi\cos\phi - \frac{\beta}{16R^3}\sin 3\phi. \tag{74}$$

(This differs slightly from Halilsoy's solution, which is missing the last term.) As $r' \to \infty$, $u' \to 0$, and $\phi \to \phi_\infty$, which may be assumed small. Thus (74) reduces to

$$0 = \frac{\phi_\infty}{R} + \frac{2m}{R^2} + \frac{9\beta\phi_\infty}{16R^3}\,.$$

The total deflection is $\Delta\phi_\infty = 2|\phi_\infty|$ or

$$\Delta\phi_\infty \approx \frac{4m}{R}\left(1 - \frac{9\beta}{16R^2}\right). \tag{75}$$

The coefficient of $\beta/R^2$ differs from the $\frac{3}{4}$ obtained by Halilsoy, the difference being the contribution of the last term in (74).

For some of the metrics under consideration [see (46) and (47)], $\beta$ can be of order $m^2$. Thus, the expressions for the light orbit (74) and for the bending (75) should be carried to higher order in $m/R$ to provide a fair assessment of the importance of the contributions due to $\beta$. This may be done by iteration. That is, we substitute (74) for $u'$ on the right-hand side of (73) and proceed as before. The result is that the following terms should be added to the right-hand side of (74)

$$\frac{-15m^2}{4R^3}\,\phi\cos\phi - \frac{3m^2}{16R^3}\sin 3\phi. \tag{76}$$

The expression (75) for the deflection of the light ray becomes

$$\Delta\phi_\infty \approx \frac{4m}{R}\left(1 - \frac{9\beta}{16R^2} + \frac{69m^2}{16R^2}\right). \tag{77}$$

*Precession of planetary apsides*

For a planet, we return to (72). This equation is exact and may be handled as it stands. However, since we will treat some of the terms on the right side of the equation as perturbations, no precision will be lost by replacing the constants of the motion $h$ and $H$ by their classical limits. For a planet moving at non-relativistic speed, we may, by (27), put $H^2 \approx c_0^4$. Also, at sufficiently large $r$ (i.e., at the radius of a planetary orbit), we may put $h \approx h_0$ the classical angular momentum per unit mass. Thus we have

$$\frac{d^2u'}{d\phi^2} + u' - \frac{mc_0^2}{h_0^2} = -\frac{\beta c_0^2}{h_0^2}\,u' + 3mu'^2 - 2\beta u'^3. \tag{78}$$

This differential equation is not quite the same as that recently obtained through other means by Halilsoy. In particular, Halilsoy's equation is missing the term $-\beta(c_0/h_0)^2u'$ [Ref. 23, eq. (5)].

If we temporarily put the terms in $u'^2$ and $u'^3$ equal to zero, we obtain a differential equation that may be solved exactly:

$$\frac{d^2u'}{d\phi^2} + s^2u' = \frac{1}{\alpha_0}, \tag{79}$$

where

$$s^2 = 1 + \beta c_0^2/h_0^2, \tag{80}$$

and

$$\alpha_0 = h_0^2/mc_0^2. \tag{81}$$

The solution is the precessing ellipse

$$u' = \alpha^{-1}(1 + e\cos s\phi), \tag{82}$$

where the eccentricity $e$ is arbitrary and where the semi-latus rectum $\alpha$ is

$$\alpha = \alpha_0\left[1 + \frac{\beta c_0^2}{h_0^2}\right]. \tag{83}$$

The precession of the apsides, per revolution of the planet on the orbit, due to the term in $\beta u'$, is then

$$\triangle_1 = \frac{-\pi\beta c_0^2}{h_0^2} = \frac{-\pi m}{\alpha_0}\frac{\beta}{m^2}. \tag{84}$$

The terms in $u'^2$ and $u'^3$ may be treated as perturbations. Thus, one inserts (82) in these two terms on the right side of (78) and solves the resulting equation. The term in $u'^2$, acting alone, produces the usual precession associated with the Schwarzschild problem:

$$\triangle_2 = \frac{6\pi m^2}{h_0^2}c_0^2 = \frac{6\pi m}{\alpha_0}. \tag{85}$$

As shown by Halilsoy, the term in $u'^3$, acting alone, produces the precession

$$\triangle_3 = -\frac{6\pi\beta m^2}{h_0^4}c_0^4 = -6\pi\left[\frac{m}{\alpha_0}\right]^2\frac{\beta}{m^2}. \tag{86}$$

However, this term is smaller than (84) by a factor of $m/\alpha_0$ and is therefore negligible. To lowest order in $m/\alpha_0$, then, the total precession $\triangle$ of the apsides per revolution is just $\triangle_1 + \triangle_2$:

$$\triangle = \frac{6\pi m}{\alpha_0}\left(1 - \frac{\beta}{6m^2}\right). \tag{87}$$

*Radar echo delay*

We consider the propagation in time of light in an RN-type metric. Again, let the motion take place in the $\theta = \pi/2$ plane. Writing out the conservation of energy equation (22) in plane polar coordinates and making use of (25) and (58), we have

$$\left(\frac{dr}{dA}\right)^2 + \frac{h^2}{r^2} - n^2 c_0^2 = 0\,. \tag{88}$$

Let us now evaluate the constant of the motion, $h$. Let $r_0$ denote the distance of closest approach of the ray to the center of the gravitating body. When $r = r_0$, we have $dr/dA = 0$. Thus (88) gives

$$h = r_0 n(r_0) c_0\,, \tag{89}$$

which is analogous to the classical-mechanical expression $r_0 v(r_0)$.

We may now transform from $r$ back to $r'$ using (41) and (42). Also, because we are interested in the propagation of light in time, we use (24) to pass over from $A$ to $t$ as independent variable. Thus, with substitution and transformation of (89), (88) becomes

$$\left(\frac{dr'}{dt}\right)^2 = \Omega^4(r')c_0^2\left[1 - \frac{r_0'^2}{r'^2}\frac{\Omega^2(r')}{\Omega^2(r_0')}\right]. \tag{90}$$

The time of travel from $r_0$ to $r'$ is then

$$\begin{aligned}\Delta t &= c_0^{-1}\int_{r_0'}^{r'} \Omega^{-2}(r')\left[1 - \frac{r_0'^2}{r'^2}\frac{\Omega^2(r')}{\Omega^2(r_0')}\right]^{-1/2} dr' \\ &\equiv c_0^{-1}\int_{r_0'}^{r'} I(r')\,dr'. \end{aligned} \tag{91}$$

Now,

$$I = \Omega^{-2}\left(1-\frac{r_0'}{r'^2}\right)^{-1/2}\left[1+\frac{[1-\Omega^2(r')/\Omega^2(r_0')]}{(r'^2/r_0'^2-1)}\right]^{-1/2}. \tag{92}$$

Using (39) to write out $\Omega(r')$ and $\Omega(r_0')$, then expanding to first order in $m$ and first order in $\beta$, we obtain

$$I = \left(1-\frac{r_0'^2}{r'^2}\right)^{-1/2}\left[1+\frac{2m}{r'}+\frac{mr_0'}{r'(r'+r_0')}-\frac{3\beta}{2r'^2}\right]. \tag{93}$$

The total time of travel $\Delta t(r_0', r')$ from $r_0'$ to $r'$ is obtained by substituting (93) into (91) and integrating:

$$\begin{aligned}\Delta t(r_0', r') \approx\ & c_0^{-1}(r'^2 - r_0'^2)^{1/2} \\ & + \frac{2m}{c_0}\ln\left[\frac{r'}{r_0'}+\frac{(r'^2-r_0'^2)^{1/2}}{r_0'}\right] \\ & + \frac{m}{c_0}\left[\frac{r'-r_0'}{r'+r_0'}\right]^{1/2} + \frac{3\beta}{2r_0'c_0}\sin^{-1}\left(\frac{r_0'}{r'}\right) - \frac{3\pi\beta}{4r_0'c_0}.\end{aligned} \tag{94}$$

The first term on the right side of (94) is the transit time of light in Euclidean space. The delay $\Delta T(r_0, r')$ due to general-relativisitic effects is the sum of the remaining terms.

As an example, let us estimate the radar echo delay for a signal sent from the Earth at radius $r_\mathrm{e}'$ to an inferior planet at radius $r_\mathrm{p}'$ when that planet is near superior conjunction with the Sun. Let the distance of closest approach to the center of the Sun be $r_0'$. If we suppose that the signal passes very near the Sun, so that $r_0'$ is much smaller than either $r_\mathrm{e}'$ or $r_\mathrm{p}'$, then

$$\Delta T(r_0', r_\mathrm{e}') \approx \frac{2m}{c_0}\ln\left[\frac{2r_\mathrm{e}'}{r_0'}\right]+\frac{m}{c_0}\left[1-\frac{r_0'}{r_\mathrm{e}'}\right]-\frac{3\pi\beta}{4c_0r_0'}\left[1-\frac{2r_0'}{\pi r_\mathrm{e}'}\right], \tag{95}$$

and the total delay in the signal for the round trip is

$$2[\Delta T(r_0', r_\mathrm{e}') + \Delta T(r_0', r_\mathrm{p}')]. \tag{96}$$

### 5.4. Redshifts

The gravitational and cosmological redshifts are not dynamical effects; i.e., we do not need to solve an equation of motion in order to calculate them. However, it may be of some interest to see how the redshifts arise in the language of an effective index of refraction.

In ordinary optics, if a light wave travels from a region of high index of refraction $n_2$ to a region of low index of refraction $n_1$, the wavelength $\lambda$ increases because the leading part of the wave enters $n_1$ first and speeds up while the trailing part is still in $n_2$. Thus the wave begins to stretch out. $\lambda$ and $c$ change but the frequency $\nu$ does not. This holds even if the index varies continuously with the spatial coordinates and even if the ray crosses obliquely through the contours of constant $n$. Thus, in general,

$$\lambda(\mathbf{r}_1)n(\mathbf{r}_1) = \lambda(\mathbf{r}_2)n(\mathbf{r}_2). \tag{97}$$

Now consider a situation in which $n$ does not depend on the spatial coordinates, but does vary with time. An example can be imagined: let the air slowly be pumped from a chamber. Then, as long as the wave does not leave the chamber, $n$ is everywhere the same, but is a decreasing function of time. The wavelength will not change, since the leading edge of the wave never encounters a new value of $n$ before the trailing edge does. Thus, in this case $\nu$ and $c$ change but $\lambda$ does not. We have

$$\lambda(t_1) = \lambda(t_2). \tag{98}$$

Suppose now that the index of refraction can be written as a product of two functions — a function $n_s$ of the spatial coordinates alone and a function $n_t$ of the time alone, as in (66). For the reasons just mentioned, $n_t$ does not affect the wavelength and we have

$$\lambda(\mathbf{r}_1)n_s(\mathbf{r}_1) = \lambda(\mathbf{r}_2)n_s(\mathbf{r}_2). \tag{99}$$

We wish to apply these rules of ordinary optics to the propagation of light in general relativity. Our effective index of refraction (3) is based upon the isotropic coordinate speed of light. Thus the quantity analogous to the wavelength of classical optics is the coordinate distance between successive crests of the wave. Coordinate distances are not, of course, directly measurable in general relativity. The physically measurable metric length is obtained from the coordinate length by means of the metric (1).

*Gravitational redshift*

Let $|\Delta\mathbf{r}_1|$ be the coordinate distance between successive crests of a light wave located at $\mathbf{r}_1$. Similarly, let $|\Delta\mathbf{r}_2|$ be the coordinate distance between successive crests at a different point $\mathbf{r}_2$ located on the same ray. Then, in analogy to the condition (97) from ordinary optics, we must have

$$|\Delta\mathbf{r}_1|n(\mathbf{r}_1) = |\Delta\mathbf{r}_2|n(\mathbf{r}_2). \tag{100}$$

The metric length $\lambda$ of the wave at point $\mathbf{r}_1$ is obtained by applying the metric (1) to the coordinate length $|\Delta\mathbf{r}_1|$ of the wave:

$$\lambda(\mathbf{r}_1) = \Phi^{-1}(\mathbf{r}_1)|\Delta\mathbf{r}_1|.$$

A similar expression holds for $\lambda(\mathbf{r}_2)$. Thus we have

$$\Phi(\mathbf{r}_1)\lambda(\mathbf{r}_1)n(\mathbf{r}_1) = \Phi(\mathbf{r}_2)\lambda(\mathbf{r}_2)n(\mathbf{r}_2), \tag{101}$$

or, using (3),

$$\lambda(\mathbf{r}_1)\Omega^{-1}(\mathbf{r}_1) = \lambda(\mathbf{r}_2)\Omega^{-1}(\mathbf{r}_2), \tag{102}$$

the usual gravitational redshift relation.

As an example, let us take the case of the Schwarzschild metric. Let a source of light be located at $\mathbf{r}_1$ and an observer at $\mathbf{r}_2$, sufficiently far from the central gravitating body so that we may put $\Omega(\mathbf{r}_2) \approx 1$. Then, with the use of (39) and (43), (102) gives

$$\begin{aligned} z &\equiv (\lambda_{\text{observed}} - \lambda_{\text{emitted}})/\lambda_{\text{emitted}} \\ &= (1 - 2m/r')^{-1/2} - 1. \end{aligned} \tag{103}$$

This result may, of course, be derived by many other methods.

*Cosmological redshift*

In the expanding universe of the Robertson–Walker metric, we have an index of refraction (57) that factors like (66), with $n_{\mathrm{s}}$ and $n_{\mathrm{t}}$ given by (67) and (68). Let a light wave of coordinate length $|\Delta\mathbf{r}_1|$ be emitted at $(t_1, \mathbf{r}_1)$ and received at $(t_2, \mathbf{r}_2)$. By analogy to (99), the coordinate length $|\Delta\mathbf{r}_1|$ of the received wave is determined by

$$|\Delta\mathbf{r}_1|n_{\mathrm{s}}(\mathbf{r}_1) = |\Delta\mathbf{r}_2|n_{\mathrm{s}}(\mathbf{r}_2). \tag{104}$$

The metric length $\lambda$ of the wave is obtained by applying the metric (56) to the coordinate length $|\Delta\mathbf{r}|$. Thus (104) becomes

$$\frac{(1 + kr_1^2/4)}{R(t_1)}\lambda(t_1, \mathbf{r}_1)n_{\mathrm{s}}(\mathbf{r}_1) = \frac{(1 + kr_2^2/4)}{R(t_2)}\lambda(t_2, \mathbf{r}_2)n_{\mathrm{s}}(\mathbf{r}_2),$$

or, with use of (67),

$$\frac{\lambda_1}{R(t_1)} = \frac{\lambda_2}{R(t_2)}, \tag{105}$$

the usual cosmological redshift relation.

While many writers have stressed the fundamentally different natures of the gravitational and cosmological redshifts, others have argued that it is possible to treat them with a single unified approach [29]. In the effective optical-medium formulation pursued here, it is interesting to note that both spectral shifts depend on a single optical principle (99).

## 6. CONCLUSION

The Newtonian forms (21) and (22) for the geodesic equations of motion offer some practical advantages for calculation. In particular, they facilitate the writing down of exact general relativistic expressions simply by analogy to classical formulas. Thus, they constitute one more tool for the relativist's tool kit. But the most interesting consequence of extending the optical-mechanical analogy to general relativity is that one simple equation of motion (21) now summarizes three fields of study: classical geometrical optics, classical particle mechanics, and geodesic motion of both light and particles in general relativity. Of course, our treatment is restricted to isotropic fields and media. Nevertheless, this unified approach, based on the use of the optical action, possesses considerable flexibility and scope. A single variational principle (14) governs all three domains.

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