Electric potential in the classical Hall effect: An unusual boundary-value problem

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The classical Hall effect presents a surprisingly unusual and challenging problem in electrostatics, with boundary conditions that are not of Dirichlet, Neumann, or of mixed Dirichlet and Neumann type. These unusual boundary conditions create several difficulties not normally encountered in standard problems, and ultimately lead to expansion of the electric potential in a nonorthogonal basis set. We derive the boundary conditions for the potential in a rectangular geometry, construct a solution for the potential, and discuss the relation between this problem and problems of the standard mixed type. We also address a commonly encountered misconception about the current distribution. © *1998 American Association of Physics Teachers.*

I. INTRODUCTION

The Hall effect was discovered over 100 years ago, and has since become a widely used experimental tool for studying the transport properties of materials, as well as the basis for a large number of technological applications.[1] One does not need to know the full electrostatic solution to the Hall problem in order to extract useful information, since voltage differences between suitable pairs of points in the current flow suffice to characterize the transport processes.[2] The full solution to the electrostatic problem is a surprisingly challenging exercise, going beyond the techniques that are most frequently used in potential theory problems.

A typical arrangement of a Hall effect experiment is illustrated in Fig. 1. A thin, rectangular metal plate lies in the $x-y$ plane. The edge at $y=H$ is maintained at electric potential V_0 and the edge at $y=-H$ is maintained at $-V_0$. Thus an electric current will flow in the $-y$ direction. If an externally produced uniform magnetic field $\mathbf{B}$ is imposed in the $-z$ direction, there will be a magnetic force on the moving (positive) charges, directed in the $+x$ direction. This gives rise to a charge separation that produces a potential gradient in the x direction. The problem, then, is to find an analytical expression for the electric potential $V(x,y)$ everywhere in the metal plate.

Although this problem is stated in simple terms, it turns out to have several unusual features. First, it is not immediately clear what conditions must be imposed on $V(x,y)$ at the boundaries $x=0$ and $x=L$. Second, once these boundary conditions are derived, they result in a boundary-value problem that is not of standard type. Usually we expect to encounter a Dirichlet problem (V specified everywhere on the boundary), a Neumann problem (normal derivative of V specified everywhere on the boundary), or a ''mixed'' problem (Dirichlet conditions on part of the boundary and Neumann on the remainder).[3] As we shall see, our simple problem falls into none of these classes. Third, as a consequence of these unusual boundary conditions we are forced to expand $V(x,y)$ in terms of nonorthogonal basis functions. Nevertheless, a series solution for $V(x,y)$ can be obtained by straightforward methods. This combination of unusual mathematical features and a simple physical situation will, we hope, make the problem of some interest to readers of this Journal. Finally, as we show in an Appendix, this problem can also be solved by reducing it to an infinite number of problems of standard type.

II. BOUNDARY CONDITIONS AT $x=0$ AND $x=L$

We would like to determine the electric potential $V(x,y)$ everywhere in the conductor. In the steady state there is no volume charge density so $\boldsymbol{\nabla}\cdot\mathbf{E}=0$ ($\mathbf{E}$ is the electric field). As this is electrostatics, $\mathbf{E}=-\boldsymbol{\nabla}V$, so we need to solve Laplace's equation,

$$\boldsymbol{\nabla}^2 V=0, \tag{1}$$

subject to the appropriate boundary conditions. The upper and lower edges of the plate are maintained at constant potential:

$$V(x,H)=V_0, \tag{2}$$

$$V(x,-H)=-V_0. \tag{3}$$

The other physical constraint is that no charge enters or leaves the sample through the sides at $x=0$ and $x=L$. We must now express this condition on the electric current in terms of the electric potential to provide us with the boundary conditions for the left- and right-hand edges.

In the case of zero magnetic field, the current density $\mathbf{J}$ and the electric field $\mathbf{E}$ are related by Ohm's law (throughout this analysis we assume linear materials)

$$\mathbf{E}=\rho\mathbf{J}, \tag{4}$$

where ρ is the resistivity of the material. Equation (4) represents a condition of balanced forces. The left side is the electrostatic force per unit charge and the right side is the negative of the drag force per unit charge. In the presence of a magnetic field, $\mathbf{B}$, the magnetic force per unit charge, $\mathbf{v}\times\mathbf{B}$, must be added to the left side of Eq. (4). The current density is related to the drift velocity, $\mathbf{v}$, as $\mathbf{J}=nq\mathbf{v}$, where q is the charge and n is the number density of the charge carriers, assumed to be constant. Thus the generalization of Eq. (4), valid with a magnetic field present, is

$$\mathbf{E}+\frac{1}{nq}\,\mathbf{J}\times\mathbf{B}=\rho\mathbf{J}. \tag{5}$$

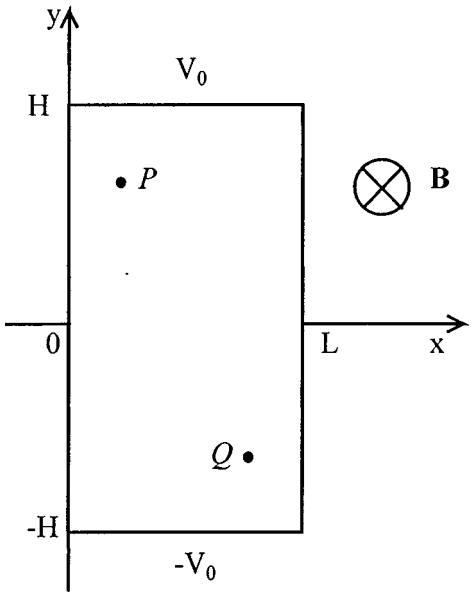

Fig. 1. Sample geometry showing the axes, top and bottom boundary conditions, and the magnetic field into the page. The points labeled P and Q are symmetry points.

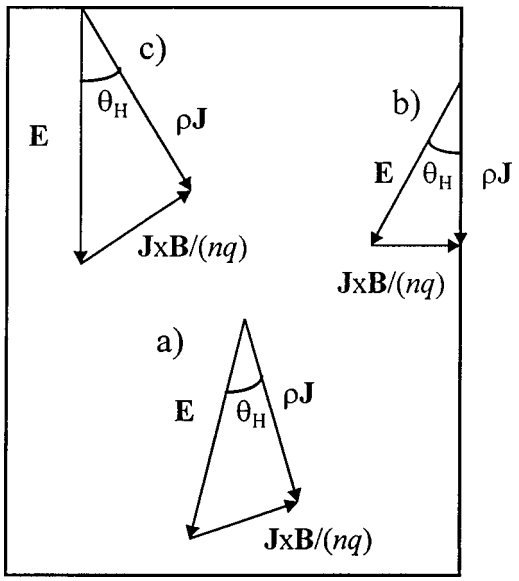

Fig. 2. Balance of forces corresponding to Eq. (5): (a) general situation, (b) at the right edge, **J** is straight down since $J_x=0$, (c) at the top edge **E** is straight down since $y=H$ is an equipotential.

Let us consider the standard problem, in which the magnetic field is perpendicular to the plane of the conducting sheet: $\mathbf{B}=-B_0\hat{\mathbf{z}}$. In general, within the sample, **J** may have both x and y components. Evaluating the cross product in Eq. (5) we obtain

$$E_x-\rho J_x-\frac{B_0}{nq}J_y=0, \tag{6}$$

$$E_y-\rho J_y+\frac{B_0}{nq}J_x=0. \tag{7}$$

Solving Eqs. (6) and (7) for the components of the current density we have[4]

$$J_x=\frac{(E_x-\lambda E_y)}{\rho(1+\lambda^2)}, \tag{8}$$

$$J_y=\frac{(E_y+\lambda E_x)}{\rho(1+\lambda^2)}, \tag{9}$$

where $\lambda=B_0/\rho nq$.

The requirement that no current leave through the edges at $x=0$ and $x=L$ means that $J_x(0,y)=0$ and $J_x(L,y)=0$. These conditions, along with Eq. (8), result in

$$E_x(0,y)=\lambda E_y(0,y), \tag{10}$$

$$E_x(L,y)=\lambda E_y(L,y). \tag{11}$$

In terms of the electric potential we have $E_x=-\partial V/\partial x$ and $E_y=-\partial V/\partial y$, so at both $x=0$ and $x=L$ we require

$$\frac{\partial V}{\partial x}=\lambda\frac{\partial V}{\partial y}. \tag{12}$$

This is the desired boundary condition on $V(x,y)$ at the left and right boundaries. *Note that the slopes of all the equipotential curves are equal to $-\lambda$ at the places where they meet the left- and right-hand edges.*

The parameter λ is also related in a simple way to a parameter called the Hall angle[5] θ_H, defined as the angle between **J** and **E**. From the vector triangle[6] of Eq. (5), illustrated in Fig. 2, we see that the Hall angle is simply

$$\theta_H=\tan^{-1}(B_0/\rho nq)=\tan^{-1}\lambda. \tag{13}$$

The magnitudes and directions of **J** and **E** may well be different at different places in the plate. Nevertheless, the angle θ_H between **J** and **E** is everywhere the same.

The vector diagram provides a more physical way of understanding the boundary conditions. Figure 2 illustrates the vector triangle at (a) an arbitrary position in the rectangle, (b) a point on the right edge, and (c) a point on the top edge. On either the left- or right-hand edges **J** must be parallel to the edge, thus **E** makes an angle θ_H with respect to the boundary [Fig. 2(b)]. This is precisely the condition expressed by Eq. (11). The top edge is an equipotential, so **E** must be perpendicular to the boundary [Fig. 2(c)].

Note that a statement frequently encountered in textbooks[7] stands in need of a correction. It is often asserted that in the steady state the magnetic force on the charge carriers just balances the horizontal component of the electric force (which is due to the Hall potential) and that, consequently, the current flows parallel to the y axis. This is so only at the left and right edges (as in Fig. 2). Elsewhere in the plate, **J** in general has an x component and the magnetic force is not entirely in the x direction. However, for the case of small magnetic field or long, thin samples, it is a reasonable approximation. We shall investigate the current distribution in Sec. VI.

III. STATEMENT OF THE BOUNDARY-VALUE PROBLEM

The mathematical problem is as follows: Find $V(x,y)$ that solves $\nabla^2V=0$ in the region $0\leqslant x\leqslant L$ and $-H\leqslant y\leqslant H$ subject to the boundary conditions:

$$V(x,H)=V_0, \tag{14}$$

$$V(x,-H)=-V_0, \tag{15}$$

$$\left.\frac{\partial V}{\partial x}\right|_{x=0}=\lambda\left.\frac{\partial V}{\partial y}\right|_{x=0}, \tag{16}$$

$$\left.\frac{\partial V}{\partial x}\right|_{x=L}=\lambda\left.\frac{\partial V}{\partial y}\right|_{x=L}. \tag{17}$$

Note that λ depends on B_0 so the ''no magnetic field'' case corresponds to $\lambda=0$.

This is an unusual set of boundary conditions. Typically, either the potential (Dirichlet), or the normal derivative of the potential (Neumann), is known on each boundary, or else the problem is of the ''mixed'' type, with Dirichlet conditions on part of the boundary and Neumann on the remainder. In our case the boundary conditions for the left and right edges are given in terms of both partial derivatives of the unknown potential, $V(x,y)$. Note that the problem reduces to the standard ''mixed'' type for $\lambda=0$. The general $(\lambda\neq 0)$ problem does not appear to be covered in the standard treatments of potential theory.[8]

IV. SOLUTION

We seek solutions of Laplace's equation that satisfy conditions (16) and (17). Any linear combination of such solutions will still satisfy (16) and (17). The appropriate linear combination can then be built up to satisfy Eqs. (14) and (15). We will handle the two forms for solutions of Laplace's equation separately.

A. Linear solution

A linear function of x or y will clearly be annihilated by the Laplacian operator. The bilinear solution $V(x,y)=(a+bx)(c+dy)$, with a,b,c,d constant, satisfies the differential equation. However, it is easy to see that this form cannot satisfy the boundary conditions Eqs. (16) and (17). Similarly $V(x,y)=(a+bx)f(y)$ or $(c+dy)g(x)$, where f and g are arbitrary functions, cannot satisfy Eqs. (16) and (17). We are therefore left with the possibility of a linear function of x plus a linear function of y. At the left or right edge, the slope of the equipotential curve is $-\lambda$. Thus the linear solution must be of the form

$$V(x,y)=a\lambda(x+b)+ay+c,$$

where a,b,c are constants. We know from elementary considerations that in the absence of a magnetic field the solution is $V(x,y)=V_0y/H$. The case of ''no magnetic field'' corresponds to $\lambda=0$ so $a=V_0/H$ and $c=0$. The linear solution is therefore of the form

$$V(x,y)=\frac{V_0}{H}\left[\lambda(x+b)+y\right]. \tag{18}$$

B. Harmonic-exponential solutions

For the nonlinear forms we assume separable solutions of the form $V(x,y)=X(x)Y(y)$. In the usual way, Laplace's equation then separates into two ordinary differential equations:

$$\frac{d^2X}{dx^2}=-k^2X, \tag{19}$$

$$\frac{d^2Y}{dy^2}=k^2Y, \tag{20}$$

where the separation constant, k^2, is yet to be determined. $X(x)$ and $Y(y)$ are, of course, real functions of their arguments.

We can easily show that k must be real, and therefore X is a trigonometric function of x and Y is an exponential function of y. To see this put $V=XY$ in Eq. (16) to find

$$X_0'Y=\lambda Y'X_0, \tag{21}$$

where the prime denotes differentiation with respect to the argument $(X'=dX/dx, Y'=dY/dy)$. The subscript 0 denotes evaluation at $x=0$. Differentiating Eq. (21) with respect to y we get

$$X_0'Y'=\lambda Y''X_0. \tag{22}$$

Substituting Eq. (20) in Eq. (22) we find

$$X_0'Y'=\lambda k^2YX_0. \tag{23}$$

Then using Eq. (21) in Eq. (23) to eliminate Y' we have

$$k^2=\left(\frac{X_0'}{\lambda X_0}\right)^2. \tag{24}$$

Consequently, k^2 is real and non-negative. Therefore, k is real and from the forms of Eqs. (19) and (20) we see that $X(x)$ is trigonometric and $Y(y)$ is exponential.

For a given value of k, the harmonic-exponential solution is then of the form

$$V_k(x,y)=(A_k\cos kx+B_k\sin kx)(C_ke^{ky}+D_ke^{-ky}). \tag{25}$$

C. Applying the boundary conditions at $x=0, L$

We now apply conditions (16) and (17) to determine the separation constant k and to restrict the range of possibilities for the constants of integration A,B,C,D. First, applying Eq. (16) to Eq. (25), we find

$$\frac{B_k}{A_k}=\lambda\frac{C_ke^{ky}-D_ke^{-ky}}{C_ke^{ky}+D_ke^{-ky}}. \tag{26}$$

Thus the right side of Eq. (26) cannot be a function of y. Hence, either

$$C_k=0, \quad B_k=-\lambda A_k, \tag{27}$$

or

$$D_k=0, \quad B_k=\lambda A_k. \tag{28}$$

For a specific value of k, the most general solution is a linear combination of both possibilities. Thus we have

$$V_k(x,y)=R_k(\cos kx+\lambda\sin kx)e^{ky}+S_k(\cos kx-\lambda\sin kx)e^{-ky}, \tag{29}$$

where $R_k\equiv A_kC_k$ (for the case $D_k=0$) and $S_k\equiv A_kD_k$ (for the case $C_k=0$).

If we impose condition (17) on Eq. (29) and note that the terms in e^{ky} and e^{-ky} must satisfy this condition separately, we find

$$(1+\lambda^2)\sin kL=0. \tag{30}$$

Hence $\sin kL=0$ and k is restricted to values k_n given by

$$k_n = n\pi/L, \tag{31}$$

where $n = 1,2,3,\ldots$.

The most general solution satisfying the boundary conditions at the left and right edges is therefore Eq. (18) plus expressions of the form of Eq. (29), with k_n restricted by Eq. (31):

$$V(x,y) = \frac{V_0}{H}[\lambda(x+b)+y] + \sum_{n=1,2,3,\ldots}[R_n(\cos k_n x + \lambda \sin k_n x)e^{k_n y} + S_n(\cos k_n x - \lambda \sin k_n x)e^{-k_n y}] \tag{32}$$

or, recombining terms,

$$V(x,y) = \frac{V_0}{H}[\lambda(x+b)+y] + \sum_{n=1,2,3,\ldots}[\cos k_n x(R_n e^{k_n y} + S_n e^{-k_n y}) + \lambda \sin k_n x(R_n e^{k_n y} - S_n e^{-k_n y})]. \tag{33}$$

D. Exploiting a symmetry

To proceed further, let us note a symmetry of the system of Eqs. (1), (14), (15), (16), and (17). These have the property that $V \to -V$ when $x \to L - x$ and $y \to -y$. This means, for example, that points P and Q in Fig. 1 have potentials with the same magnitude but opposite sign. Imposing this symmetry on Eq. (33) requires that $b = -L/2$. Furthermore, both the cosine and sine terms in the sum of Eq. (33) must satisfy the symmetry separately. (Remember that λ is a free parameter.) Now, since $k_n = n\pi/L$,

$$\cos k_n(L-x) = (-1)^n \cos k_n x,$$

$$\sin k_n(L-x) = (-1)^{n+1} \sin k_n x.$$

Requiring that each term in Eq. (33) satisfy the symmetry $V \to -V$ when $x \to L - x$ and $y \to -y$ then leads to (from the cosine terms)

$$(-1)^n(R_n e^{-k_n y} + S_n e^{k_n y}) = -R_n e^{k_n y} - S_n e^{-k_n y},$$

and to (from the sine terms)

$$(-1)^{n+1}(R_n e^{-k_n y} - S_n e^{k_n y}) = -R_n e^{k_n y} + S_n e^{-k_n y}.$$

Adding these two equations, we find

$$S_n = (-1)^{n+1} R_n. \tag{34}$$

Two cases arise:

$$\begin{aligned} &\text{for odd } n, \quad S_n = R_n, \\ &\text{for even } n, \quad S_n = -R_n. \end{aligned} \tag{35}$$

Using Eq. (35) in Eq. (33) we arrive at our final form of the solution:

$$V(x,y) = \frac{V_0}{H}\left[\lambda\left(x - \frac{L}{2}\right) + y\right] + \sum_{m=1,3,\ldots} T_m\left[\cos\left(\frac{m\pi}{L}x\right)\cosh\left(\frac{m\pi}{L}y\right) + \lambda \sin\left(\frac{m\pi}{L}x\right)\sinh\left(\frac{m\pi}{L}y\right)\right] + \sum_{n=2,4,\ldots} U_n\left[\cos\left(\frac{n\pi}{L}x\right)\sinh\left(\frac{n\pi}{L}y\right) + \lambda \sin\left(\frac{n\pi}{L}x\right)\cosh\left(\frac{n\pi}{L}y\right)\right], \tag{36}$$

where $T_m = 2R_m$ (for m odd) and $U_n = 2R_n$ (for n even). From here on we shall use the subscript m to label the odd coefficients $T_1, T_3, \ldots$ and the subscript n to label the even coefficients $U_2, U_4, \ldots$.

E. Applying the boundary conditions at $y = \pm H$

We now require that the T_m and U_n be chosen so that $V(x,y)$ satisfies the top and bottom boundary conditions $V(x, \pm H) = \pm V_0$. Thus we require

$$\begin{aligned} \pm V_0 = &\frac{V_0}{H}\left[\lambda\left(x - \frac{L}{2}\right) \pm H\right] \\ &+ \sum_{m=1,3,\ldots} T_m\left[\cos\left(\frac{m\pi}{L}x\right)\cosh\left(\frac{m\pi}{L}H\right) \pm \lambda \sin\left(\frac{m\pi}{L}x\right)\sinh\left(\frac{m\pi}{L}H\right)\right] \\ &+ \sum_{n=2,4,\ldots} U_n\left[\pm\cos\left(\frac{n\pi}{L}x\right)\sinh\left(\frac{n\pi}{L}H\right) + \lambda \sin\left(\frac{n\pi}{L}x\right)\cosh\left(\frac{n\pi}{L}H\right)\right]. \end{aligned} \tag{37}$$

Adding and subtracting the two versions of Eq. (37) gives

$$\begin{aligned} -\frac{V_0}{H}&\left[\lambda\left(x - \frac{L}{2}\right)\right] \\ &= \sum_{m=1,3,\ldots} T_m \cos\left(\frac{m\pi}{L}x\right)\cosh\left(\frac{m\pi}{L}H\right) \\ &\quad + \lambda \sum_{n=2,4,\ldots} U_n \sin\left(\frac{n\pi}{L}x\right)\cosh\left(\frac{n\pi}{L}H\right) \end{aligned} \tag{38}$$

and

$$\begin{aligned} 0 = &\lambda \sum_{m=1,3,\ldots} T_m \sin\left(\frac{m\pi}{L}x\right)\sinh\left(\frac{m\pi}{L}H\right) \\ &+ \sum_{n=2,4,\ldots} U_n \cos\left(\frac{n\pi}{L}x\right)\sinh\left(\frac{n\pi}{L}H\right). \end{aligned} \tag{39}$$

In Eqs. (38) and (39) we have on the left-hand side a function of x and on the right side its expansion in a series of sines and cosines of multiples of $\pi x/L$. *It is important to note that these functions of x do not form an orthogonal basis on the interval* $0 \leqslant x \leqslant L$. The functions $\cos(m\pi x/L)$ and $\cos(m'\pi x/L)$ are orthogonal (for $m \neq m'$) on the interval $0 \leqslant x \leqslant L$. Similarly, $\sin(n\pi x/L)$ and $\sin(n'\pi x/L)$ are orthogo-

nal. But $\sin(n\pi x/L)$ and $\cos(m\pi x/L)$ (with n even and m odd) are not. These functions are orthogonal on the interval $0\leqslant x\leqslant 2L$, but are not orthogonal on the interval $0\leqslant x\leqslant L$. This is another unusual feature of this simple Hall effect problem. The nonorthogonality of the basis functions results from the boundary conditions Eqs. (16) and (17). We must be careful in evaluating the expansion coefficients.

(To understand the nonorthogonality an analogy might be helpful. Consider writing a vector in terms of a particular set of nonorthogonal basis vectors and then determining the components. Let a vector in the plane, **A**, be written in terms of a pair of unit vectors $\hat{u}$ and $\hat{v}$ which are at an angle α. We can write $\mathbf{A}=A_u\hat{u}+A_v\hat{v}$. To find the coefficient we take the dot product of **A** with the appropriate basis vector. For example to find A_u we have $\hat{u}\cdot\mathbf{A}=A_u\hat{u}\cdot\hat{u}+A_v\hat{u}\cdot\hat{v}=A_u+A_v\cos\alpha$. We do not get just the coefficient A_u; A_v is now mixed in as well. The presence of the other coefficient is due to the lack of orthogonality. For the usual case with orthogonal basis vectors $\alpha=\pi/2$ and the second term vanishes.)

Multiply Eq. (38) by $\cos(m'\pi x/L)$ and integrate from $x=0$ to $x=L$, with the result

$$T_m=\frac{4V_0L\lambda}{\pi^2Hm^2\cosh(m\pi H/L)}-\frac{4\lambda}{\pi\cosh(m\pi H/L)}\sum_{n=2,4,\ldots}U_n\cosh\left(\frac{n\pi H}{L}\right)\frac{n}{n^2-m^2}. \tag{40}$$

Multiply Eq. (39) by $\cos(n'\pi x/L)$ and integrate from $x=0$ to $x=L$ to obtain

$$U_n=\frac{-4\lambda}{\pi\sinh(n\pi H/L)}\sum_{m=1,3,\ldots}T_m\sinh\left(\frac{m\pi H}{L}\right)\frac{m}{m^2-n^2}. \tag{41}$$

These are our conditions on the coefficients, T_m and U_n. Note that they involve coupled infinite sums. The odd coefficients, T_m, are expressed in terms of the even ones, U_n, and vice versa. This results from the failure of our basis functions to be orthogonal.

The solution for the potential $V(x,y)$ is therefore given by Eq. (36) with T_m and U_n determined by Eqs. (40) and (41).

F. Solution to order λ^2

For typical metals in experimental situations[9] $\lambda\approx 0.005-0.1$. This suggests that Eqs. (40) and (41) can be solved iteratively. Looking at Eq. (40) we see that to first order in λ we need only the first term. The second term has an explicit λ and depends on the U_n, all of which involve λ. Therefore, to first order in λ the T_m are

$$T_m\simeq\frac{4V_0L\lambda}{\pi^2Hm^2\cosh(m\pi H/L)}. \tag{42}$$

Substituting Eq. (42) into Eq. (41), we find

$$U_n\simeq\frac{-16V_0L\lambda^2}{\pi^3H\sinh(n\pi H/L)}\sum_{m=1,3,\ldots}\frac{\tanh(m\pi H/L)}{m(m^2-n^2)}. \tag{43}$$

The potential $V(x,y)$ is therefore given by Eq. (36) with T_m and U_n given by Eqs. (42) and (43). This solution, with the coefficients cut off as in Eqs. (42) and (43), is correct through order λ^2, which is certainly adequate for comparison with most experimental results.

G. Solution to any order in λ

The solution to any order in λ may be obtained by iteration with Eqs. (40) and (41). A more systematic approach is as follows. Let **T** denote a vector formed from the T_m, and **U** a vector formed from the U_n:

$$\mathbf{T}\equiv\begin{pmatrix}T_1\\0\\T_3\\0\\T_5\\\vdots\end{pmatrix},\quad \mathbf{U}\equiv\begin{pmatrix}0\\U_2\\0\\U_4\\0\\\vdots\end{pmatrix}. \tag{44}$$

Then Eqs. (40) and (41) may be written

$$\mathbf{T}=\mathbf{T}^0+\Theta\mathbf{U}, \tag{45}$$

$$\mathbf{U}=\Omega\mathbf{T}, \tag{46}$$

where the components of $\mathbf{T}^0$ are

$$T_m^0=\begin{cases}\dfrac{4V_0L\lambda}{\pi^2Hm^2\cosh(m\pi H/L)} & \text{for } m \text{ odd}\\ 0 \quad \text{otherwise.}\end{cases} \tag{47}$$

The matrix elements of Θ are

$$\Theta_{mn}=\begin{cases}\dfrac{4\lambda\cosh(n\pi H/L)}{\pi\cosh(m\pi H/L)}\dfrac{n}{n^2-m^2} & \text{for } m \text{ odd and } n \text{ even}\\ 0 \quad \text{otherwise,}\end{cases} \tag{48}$$

and the matrix elements of Ω are

$$\Omega_{nm}=\begin{cases}\dfrac{-4\lambda\sinh(m\pi H/L)}{\pi\sinh(n\pi H/L)}\dfrac{m}{m^2-n^2} & \text{for } m \text{ odd and } n \text{ even}\\ 0 \quad \text{otherwise.}\end{cases} \tag{49}$$

Substituting Eq. (46) into Eq. (45), we have

$$\mathbf{T}=\mathbf{T}^0+\Theta\Omega\mathbf{T} \tag{50}$$

or

$$(I-\Theta\Omega)\mathbf{T}=\mathbf{T}^0, \tag{51}$$

where I is the identity matrix. Thus the solution for **T** is

$$\mathbf{T}=(I-\Theta\Omega)^{-1}\mathbf{T}^0, \tag{52}$$

$$\mathbf{T}=(I+\Theta\Omega+(\Theta\Omega)^2+(\Theta\Omega)^3+\cdots)\mathbf{T}^0. \tag{53}$$

Now, as Eqs. (48) and (49) show, both Θ and Ω are proportional to λ. Equation (47) shows that $\mathbf{T}^0$ is also proportional to λ. Hence **T** only contains elements with odd powers of λ. And thus, by Eq. (46), **U** contains only even powers of λ. The truncated solution (to order λ^2) given above, with the T_m and U_n given by Eqs. (42) and (43) is equivalent to

$$\mathbf{T}\simeq\mathbf{T}^0, \tag{54}$$

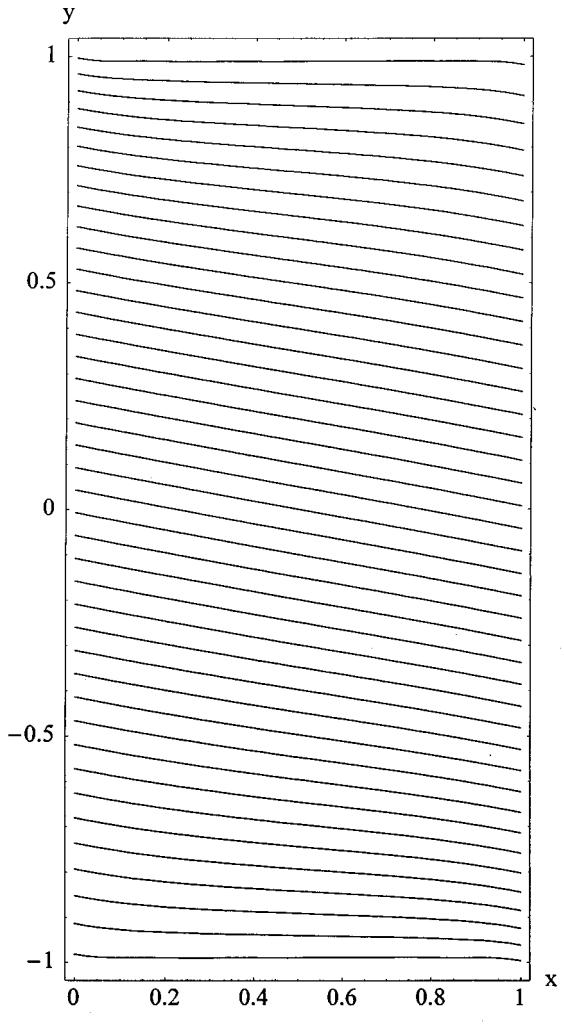

Fig. 3. Equipotential curves over the whole conductor. From top to bottom the contours shown correspond to $V=1.0,0.95,\ldots,-0.95,-1.0$. The potential was determined as discussed in the text with $\lambda=0.2$, $V_0=1$, $H=1$, $L=1$.

$$\mathbf{U}\simeq\Omega\mathbf{T}^0. \tag{55}$$

Although for real experimental situations λ is a small parameter, in principle one may solve the problem by matrix inversion, using Eq. (52), even for large λ. In this case, one must truncate the matrices to a finite number of elements in advance.

V. NUMERICAL RESULTS

To get a physical sense for our solution we evaluate the potential and the current density numerically. The solution is given by Eq. (36) and the expressions for the coefficients, T_n and U_m, are given by Eqs. (52) and (46), respectively. As the coefficients are infinite in number we truncated the vectors by taking 20 elements in both **T** and **U** and the corresponding Θ and Ω were 20×20 matrices. The sums in Eq. (36) went to $m=19$ and $n=20$, respectively. Reasonable values of the parameters were chosen to show representative behavior. In what follows we let $\lambda=0.2$, $V_0=1$, $H=1$, and $L=1$. These last three parameters just set the scale and are in arbitrary units.

Equipotential curves, $V=$const, for the conductor are shown in Fig. 3 for $V=1,0.95,\ldots,-0.95,-1$. For the case of no magnetic field the equipotentials are horizontal lines. With a magnetic field the equipotentials are clearly curved. The equipotentials get closer together as one approaches the upper left and lower right corners. This corresponds to a stronger electric field in these regions. However, as expected from the boundary conditions, the equipotentials intersect the left and right edges at a constant angle, independent of y. This can be seen in the close-up of a portion of the conductor, corresponding to the upper part, $0.8\leqslant y\leqslant 1$, as shown in Fig. 4. From top to bottom the contours correspond to $V=0.99, 0.98, 0.97,\ldots$.

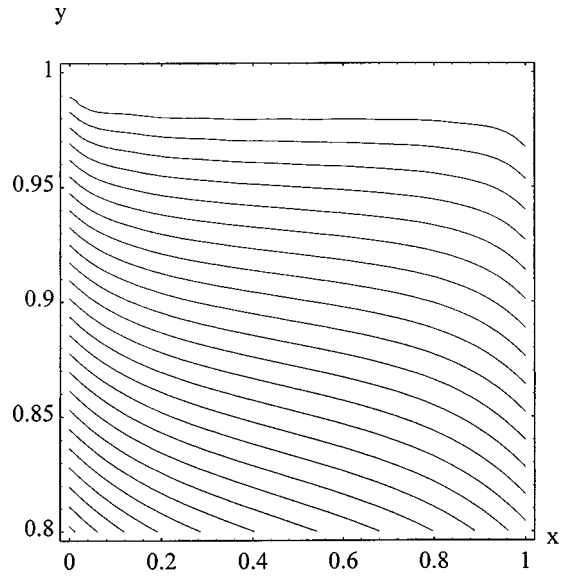

Fig. 4. Equipotential curves for the upper portion of the sample; this is an enlargement of Fig. 3 with the same parameters. Top to bottom the contours shown correspond to $V=0.99,0.98,0.97,\ldots$.

VI. THE CURRENT DISTRIBUTION

With Eq. (36) in hand for the potential, it is straightforward to investigate the current distribution. Using Eqs. (8) and (9) and taking appropriate partial derivatives of V to get the components of **E** we find

$$J_x=\frac{1}{\rho}\sum_{m=1,3,\ldots}T_m\frac{m\pi}{L}\sin(m\pi x/L)\cosh(m\pi y/L)+\frac{1}{\rho}\sum_{n=2,4,\ldots}U_n\frac{n\pi}{L}\sin(n\pi x/L)\sinh(n\pi y/L), \tag{56}$$

$$J_y=-\frac{V_0}{H\rho}-\frac{1}{\rho}\sum_{m=1,3,\ldots}T_m\frac{m\pi}{L}\cos(m\pi x/L)\sinh(m\pi y/L)-\frac{1}{\rho}\sum_{n=2,4,\ldots}U_n\frac{n\pi}{L}\cos(n\pi x/L)\cosh(n\pi y/L). \tag{57}$$

It is possible to perform two easy tests to verify that **J** as calculated is consistent with the original boundary conditions.

First, and most obviously, J_x vanishes at $x=0$ and $x=L$. This is consistent with the boundary condition illustrated by Fig. 2(b).

Second, we shall show that at $y=\pm H$, $J_x=-\lambda J_y$. This means that at the top or bottom edge, the current makes an angle $\theta_H=\tan^{-1}\lambda$ with the normal to the boundary, as shown in Fig. 2(c). That this condition is met by Eqs. (56) and (57) is not immediately obvious. Therefore, differentiate Eq. (37) with respect to x and divide by ρ to obtain

$$0=\frac{V_0\lambda}{\rho H}+\sum_{m=1,3,\ldots}\frac{1}{\rho}T_m\frac{m\pi}{L}\left[-\sin\left(\frac{m\pi}{L}x\right)\cosh\left(\frac{m\pi}{L}H\right)\pm\lambda\cos\left(\frac{m\pi}{L}x\right)\sinh\left(\frac{m\pi}{L}H\right)\right]+\sum_{n=2,4,\ldots}\frac{1}{\rho}U_n\frac{n\pi}{L}\left[\mp\sin\left(\frac{n\pi}{L}x\right)\sinh\left(\frac{n\pi}{L}H\right)+\lambda\cos\left(\frac{n\pi}{L}x\right)\cosh\left(\frac{n\pi}{L}H\right)\right]. \tag{58}$$

Evaluate Eq. (56) at $y=\pm H$, and to the resulting equation add Eq. (58) to obtain

$$J_x(x,\pm H)=\frac{V_0\lambda}{\rho H}\pm\frac{\lambda}{\rho}\sum_{m=1,3,\ldots}T_m\frac{m\pi}{L}\cos(m\pi x/L)\times\sinh(m\pi H/L)+\frac{\lambda}{\rho}\sum_{n=2,4,\ldots}U_n\frac{n\pi}{L}\times\cos(n\pi x/L)\cosh(n\pi H/L). \tag{59}$$

Comparing Eq. (59) with Eq. (57) evaluated at $y=\pm H$, we see indeed that

$$J_x(x,\pm H)=-\lambda J_y(x,\pm H) \tag{60}$$

for all x. Thus all across the top boundary, the current enters at an angle θ_H from the normal.

It is worthwhile to note one final point about the y component of the current. Integrating Eq. (57) across a horizontal section gives the net current in the y direction:

$$I_y=t\int_0^L J_y dx=\frac{-V_0Lt}{\rho H}, \tag{61}$$

where t is the thickness of the conducting plate. This is just the current we would expect in the absence of a magnetic field. This is why the standard elementary treatments, which ignore the existence of a transverse current, nevertheless obtain reasonable behavior. Although the current is not everywhere in the y direction, the net y current does not depend on the magnetic field.

To obtain numerical results for the current density we use the same procedure for determining the coefficients T_m and U_n as described above for the potential calculation. The same parameters were also used, $\lambda=0.2$, $V_0=1$, $H=1$, and $L=1$. In addition we have let $\rho=1$. (This just sets the overall scale and we are interested in relative behavior.) We evaluate Eqs. (56) and (57) with the coefficients T_m and U_n given by Eqs. (52) and (46). In Fig. 5 we show the vector field for the current density on a uniform grid of points. The direction of the current density, **J**, at a point is determined by the x and y components. The relative magnitude of the current density is indicated by the length. Along the top edge the magnitude of the current varies with position x; however, the current enters the sample with the same angle everywhere along the top and leaves with the same angle everywhere along the bottom. Notice that this implies that more charge enters the upper left than the upper right. Correspondingly, more charge leaves the lower right than the lower left. Note that there is an overall flow of charge from left to right. Also shown are three ''streamlines.'' These flow lines are everywhere tangent to the local **J** and correspond to paths of individual charges. Charge entering along the left edge continues straight down to the bottom. Charge entering in the central portion of the top edge follows a curved path with a net transport in the x direction. The amount of net left to right deflection is the greatest for charges entering at the center of the top edge.

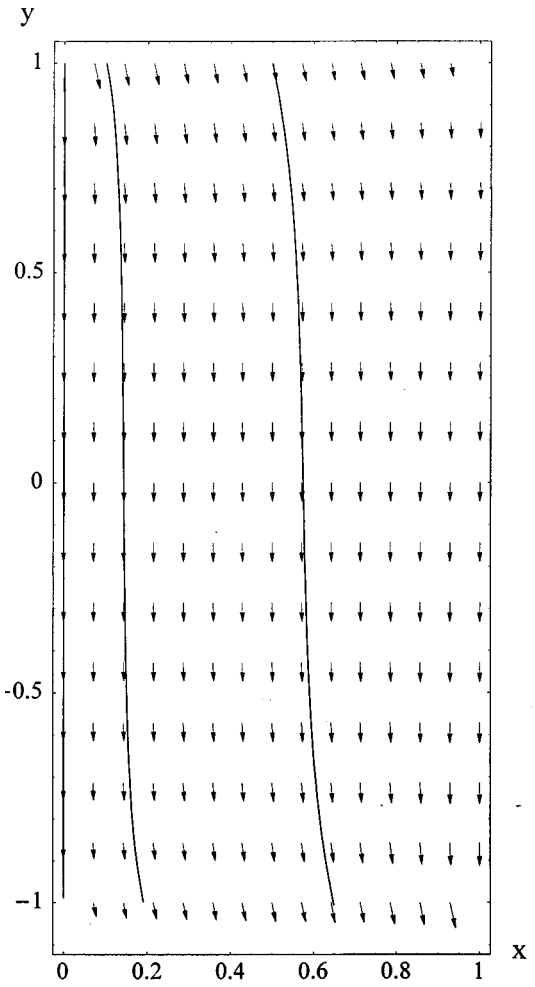

Fig. 5. Vector field of the current density on a uniform grid. The arrows indicate relative magnitude and direction. The three solid lines indicate flow lines and correspond to the mean path of a charge. (The flow line is everywhere tangent to the local **J**.) Calculated as described in the text with $\lambda=0.2$, $V_0=1$, $H=1$, $L=1$.

VII. CONCLUSION

Most textbooks of electromagnetic theory warn the reader that if one abandons the standard classes of boundary conditions (Dirichlet, Neumann, or ''mixed''), the solubility of Laplace's equation is no longer a foregone conclusion. How-

ever, physically plausible examples of nonstandard problems appear to be rather few in number. In this paper we have presented a solution to a nonstandard boundary value problem that arises naturally in the context of a surprisingly simple Hall effect situation.

ACKNOWLEDGMENTS

We thank Alan Thorndike and Fred Slee for stimulating discussions. We are grateful for suggestions by an anonymous reviewer which have improved the clarity of the presentation.

APPENDIX I: REDUCTION OF THE PROBLEM TO AN INFINITE NUMBER OF PROBLEMS OF STANDARD TYPE

Because our boundary-value problem is not of standard type, it is not immediately obvious from a mathematical point of view why it should be soluble. It is therefore worth investigating the relation between our problem and problems of the standard type. To do so, we shall expand the potential in powers of the parameter λ and then successively solve it to each order. Our problem then reduces to an infinite set of ''standard'' problems where the potential or its normal derivative is known on the boundaries. Essentially one solves the problem to some order in λ and uses that solution as input for the next higher order.

Consider Laplace's equation $\nabla^2 V=0$ with the solution written

$$V(x,y)=V^{(0)}(x,y)+\lambda V^{(1)}(x,y)+\lambda^2 V^{(2)}(x,y)+\cdots, \tag{62}$$

where superscripts indicate the corresponding order of λ for each term. Now the boundary conditions are $V(x,\pm H)=\pm V_0$ and $\partial V/\partial x=\lambda\,\partial V/\partial y$ at $x=0$, $x=L$. It is instructive to see how this works for the first few terms,

$$\begin{aligned}\pm V_0 &= V^{(0)}(x,\pm H)+\lambda V^{(1)}(x,\pm H)\\ &\quad +\lambda^2 V^{(2)}(x,\pm H)+\cdots\end{aligned} \tag{63}$$

and

$$\begin{aligned}&\left[\frac{\partial V^{(0)}}{\partial x}+\lambda\,\frac{\partial V^{(1)}}{\partial x}+\lambda^2\,\frac{\partial V^{(2)}}{\partial x}+\cdots\right]_{x=0,L}\\ &\quad=\lambda\left[\frac{\partial V^{(0)}}{\partial y}+\lambda\,\frac{\partial V^{(1)}}{\partial y}+\lambda^2\,\frac{\partial V^{(2)}}{\partial y}+\cdots\right]_{x=0,L}.\end{aligned} \tag{64}$$

From Eq. (63) we equate the coefficients of successive powers of λ on each side to obtain

$$\begin{aligned}&V^{(0)}(x,\pm H)=\pm V_0,\\ &V^{(1)}(x,\pm H)=0,\\ &V^{(2)}(x,\pm H)=0,\ldots.\end{aligned} \tag{65}$$

Matching powers of λ in Eq. (64) we see

$$\begin{aligned}&\left.\frac{\partial V^{(0)}}{\partial x}\right|_{x=0,L}=0,\\ &\left.\frac{\partial V^{(1)}}{\partial x}\right|_{x=0,L}=\left.\frac{\partial V^{(0)}}{\partial y}\right|_{x=0,L},\\ &\left.\frac{\partial V^{(2)}}{\partial x}\right|_{x=0,L}=\left.\frac{\partial V^{(1)}}{\partial y}\right|_{x=0,L},\ldots.\end{aligned} \tag{66}$$

Notice that, for each order in λ, the derivative at the left and right boundaries, $\partial V/\partial x$, is now a known function of y, given in terms of the derivative of the solution at the previous order.

$V^{(0)}(x,y)$ solves Laplace's equation subject to the boundary conditions

$$\begin{aligned}&V^{(0)}(x,\pm H)=\pm V_0 \quad \text{(Dirichlet on top and bottom)},\\ &\left.\frac{\partial V^{(0)}}{\partial x}\right|_{x=0,L}=0 \quad \text{(Neumann on left and right sides)}.\end{aligned} \tag{67}$$

This can be done by inspection and yields $V^{(0)}(x,y)=V_0 y/H$. Note this is just the $\lambda=0$ part of our solution, Eq. (36).

$V^{(1)}(x,y)$ solves Laplace's equation subject to the boundary conditions

$$\begin{aligned}&V^{(1)}(x,\pm H)=0 \quad \text{(Dirichlet on top and bottom)}\\ &\left.\frac{\partial V^{(1)}}{\partial x}\right|_{x=0,L}=\left.\frac{\partial V^{(0)}}{\partial y}\right|_{x=0,L} \quad \text{(Neumann on left and right sides)}\\ &\qquad=\text{known function of } y\\ &\qquad=V_0/H.\end{aligned} \tag{68}$$

Solving this problem by standard methods yields

$$\begin{aligned}V^{(1)}(x,y)=&\sum_{m'=1,3,\ldots}A^{(1)}_{m'}\sinh\left[\frac{m'\pi}{2H}\left(x-\frac{L}{2}\right)\right]\\ &\times\cos\left(\frac{m'\pi}{2H}\,y\right),\end{aligned} \tag{69}$$

where

$$A^{(1)}_{m'}=\frac{-8V_0}{m'^2\pi^2}\,\frac{(-1)^{(m'+1)/2}}{\cosh(m'\pi L/4H)}. \tag{70}$$

This may be compared with the order-λ terms of the solution given above, in Eqs. (36) and (42). Denote the order-λ part of this solution $\lambda V_1(x,y)$. Then from Eq. (36) we have

$$\begin{aligned}V_1(x,y)=&\frac{V_0}{H}\left(x-\frac{L}{2}\right)\\ &+\sum_{m=1,3,\ldots}T^{(1)}_m\cos\left(\frac{m\pi}{L}\,x\right)\cosh\left(\frac{m\pi}{L}\,y\right),\end{aligned} \tag{71}$$

where the $T^{(1)}_m$ are as given by Eq. (42):

$$T^{(1)}_m=\frac{4V_0L}{\pi^2 H m^2\cosh(m\pi H/L)}.$$

As Eqs. (69) and (71) do not look much alike, it is worthwhile to show that they are exactly equivalent.

To see this, let us perform the following expansions. Expand $\sinh[m'\pi(x-L/2)/2H]$ from Eq. (69) in terms of the functions $\cos(m\pi x/L)$, on the interval $0\leqslant x\leqslant L$, where $m=1,3,5,\ldots$:

$$\sinh\left[\frac{m'\pi}{2H}\left(x-\frac{L}{2}\right)\right]=-\frac{8m'LH}{\pi}\sum_{m=1,3,\ldots}\frac{\cosh(m'\pi L/4H)}{m'^2L^2+4m^2H^2}\cos\left(\frac{m\pi}{L}x\right). \tag{72}$$

Similarly, expanding $x-L/2$ from Eq. (71) on the interval $0\leqslant x\leqslant L$ in terms of the same $\cos(m\pi x/L)$ yields

$$x-\frac{L}{2}=-\frac{4L}{\pi^2}\sum_{m=1,3,\ldots}\frac{1}{m^2}\cos\left(\frac{m\pi}{L}x\right). \tag{73}$$

Finally, expand $\cosh(m\pi y/L)$ from Eq. (71) on the interval $-H\leqslant y\leqslant H$ in terms of the $\cos(m'\pi y/2H)$, where $m'=1,3,5,\ldots$:

$$\cosh\left(\frac{m\pi}{L}y\right)=\cosh\left(\frac{m\pi H}{L}\right)\left[1+\frac{16m^2H^2}{\pi}\times\sum_{m'=1,3,\ldots}\frac{(-1)^{(m'+1)/2}}{m'(m'^2L^2+4m^2H^2)}\times\cos\left(\frac{m'\pi}{2H}y\right)\right]. \tag{74}$$

If we substitute Eqs. (73) and (74) into Eq. (71) we obtain

$$V_1(x,y)=\frac{64V_0LH}{\pi^3}\sum_{m=1,3,\ldots}\sum_{m'=1,3,\ldots}\frac{(-1)^{(m'+1)/2}}{m'(m'^2L^2+4m^2H^2)}\times\cos\left(\frac{m\pi}{L}x\right)\cos\left(\frac{m'\pi}{2H}y\right). \tag{75}$$

And if we substitute Eq. (72) into Eq. (69), we obtain an expression for $V^{(1)}(x,y)$ which is identical to the right side of Eq. (75). Thus Eq. (69) is indeed equivalent to the first-order part of Eq. (36).

The sequence of problems beginning with Eqs. (67) and (68) may be continued to whatever order is necessary. This reduces our original problem (posed in Sec. III) to an infinite set of standard problems with ''mixed'' boundary conditions.[10] From a mathematical point of view, this explains why our nonstandard problem should indeed have a stable, unique solution. From a practical point of view, however, the solution of this problem by reduction to a sequence of standard problems is infinitely more tedious!

APPENDIX II: ACHIEVING A UNIFORM CURRENT DENSITY

The complexity of the potential problem is drastically reduced if we adopt a different geometry. Let the conducting plate be a parallelogram, as in Fig. 6. Two edges (1 and 2) are parallel to the x axis. The other two edges (3 and 4) make an angle with the y axis that is equal to the Hall angle $\theta_H=\tan^{-1}\lambda=\tan^{-1}(B_0/\rho nq)$. Thus the geometry of the sample must be preselected to match the material properties (ρ and n) and the externally imposed magnetic field (B_0).

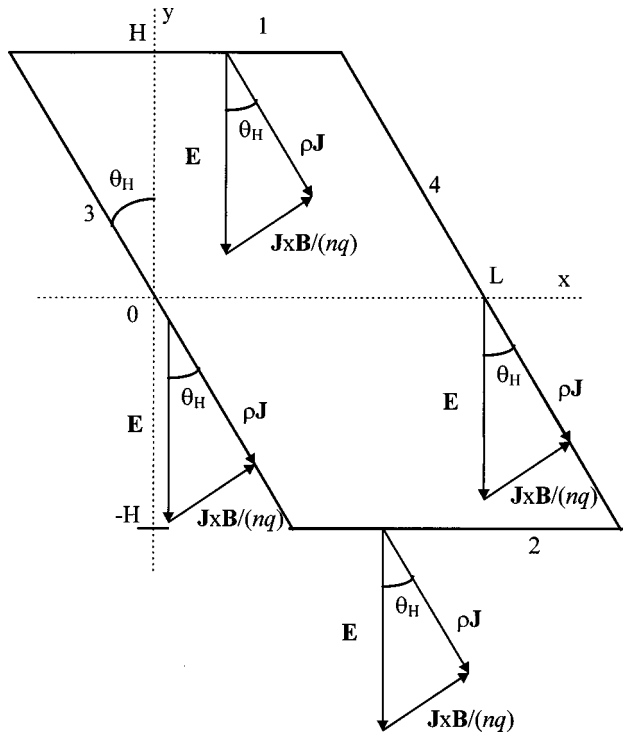

Fig. 6. Specially chosen parallelogram geometry for the Hall effect as described in Appendix II. The force triangle corresponding to Eq. (5) is shown on all four boundaries.

Note that now the triangle of forces takes the same orientation on all four boundaries. This means that the boundary condition for edges 3 and 4 is now $E_x=0$. We must solve Laplace's equation subject to the boundary conditions

$$V(x,H)=V_0 \quad \text{(edge 1)} \tag{76}$$

$$V(x,-H)=-V_0 \quad \text{(edge 2)} \tag{77}$$

$$\frac{\partial V}{\partial x}=0 \quad \text{(edges 3 and 4)}. \tag{78}$$

This is not a problem of the standard type, but in this case the solution is trivial:

$$V(x,y)=\frac{V_0}{H}y \tag{79}$$

everywhere inside the parallelogram. The equipotentials are horizontal lines.

From Eqs. (8) and (9),

$$J_x=\frac{\lambda V_0}{H\rho(1+\lambda^2)} \tag{80}$$

$$J_y=\frac{-V_0}{H\rho(1+\lambda^2)}\,. \tag{81}$$

Thus **J** is everywhere constant and parallel to edge 3. The magnitude of **J** is

$$J=\frac{V_0}{H\rho\sqrt{1+\lambda^2}}\,. \tag{82}$$

The combination $2H\sqrt{1+\lambda^2}$ is, of course, just the slant height of the parallelogram, i.e., the length of edge 3, which

is the actual distance that a charge must travel through the material.

The total current I is obtained by multiplying J by the cross-sectional area of the sample (taken perpendicular to **J**), $tL\ \cos\theta_H = tL/\sqrt{1+\lambda^2}$. Thus

$$I = \frac{tLV_0}{H\rho(1+\lambda^2)}\,, \tag{83}$$

which may be compared to Eq. (61). Although the surface areas of the plates shown in Figs. 1 and 6 are the same $(2HL)$, the direction of current flow in Fig. 6 produces an effective increase in length, and a decrease in width, of the conductor.

So, another way to express the limitations of the traditional elementary treatment is to say that it blurs the distinction between the geometries of Figs. 1 and 6. In the limit of long, thin samples and low magnetic fields the two are approximately the same. It is possible that the geometry of Fig. 6 may have some utility in experimental applications.

[1]*The Hall Effect and Its Applications*, edited by C. L. Chien and C. R. Westgate (Plenum, New York, 1976).

[2]E. H. Putley, *The Hall Effect and Semiconductor Physics* (Dover, New York, 1968).

[3]See J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), 2nd ed., pp. 42, 121. ''Mixed'' can also refer to the case when a linear combination of the potential and its normal derivative is specified along the boundary, see Daniel Zwillinger, *Handbook of Differential Equations* (Academic, San Diego, 1989), p. 2.

[4]This derivation can be found in Paul Lorrain, Dale R. Corson, and François Lorrain, *Electromagnetic Fields and Waves* (Freeman, San Francisco, 1988), 3rd ed., p. 408, and Niel W. Ashcroft and N. David Mermin, *Solid State Physics* (Saunders College Press, Philadelphia, 1976), pp. 11–14.

[5]See Ref. 2, p. 24, and Gregory H. Wannier, *Elements of Solid State Theory* (Cambridge U.P., New York, 1959), p. 195.

[6]The force triangle is discussed by A. B. Pippard, *Magnetoresistance in Metals* (Cambridge U.P., New York, 1989), p. 4.

[7]The following treatments, which are typical, explicitly assume that the current flow is one dimensional. (This is a reasonable assumption for small magnetic fields and long thin samples.) P. M. Fishbane, S. Gasiorowicz, and S. T. Thornton, *Physics for Scientists and Engineers* (Prentice Hall, Englewood Cliffs, NJ, 1996), 2nd ed., pp. 795–796; Charles Kittel, *Introduction to Solid State Physics* (Wiley, New York, 1996), 7th ed., pp. 164–166; Richard H. Bube, *Electrons in Solids* (Academic, New York, 1992), 3rd ed., pp. 196–199.

[8]See, for example, Philip M. Morse and Herman Feshbach, *Methods of Theoretical Physics* (McGraw–Hill, New York, 1953); Ian N. Sneddon, *Mixed Boundary Value Problems in Potential Theory* (Wiley, New York, 1966); George Arfken, *Mathematical Methods for Physicists* (Academic, San Diego, 1985), 3rd ed.

[9]See Charles Kittel, in Ref. 7. For copper (pp. 24 and 160): $q = 1.60 \times 10^{-19}$ C, $n = 8.45 \times 10^{28}$ m^{-3}, $\rho = 1.70 \times 10^{-8}\ \Omega$ m, so $\lambda = 4.35 \times 10^{-3} B_0$ with B_0 in tesla. So for a typical field of roughly 1–20 T we get $\lambda \approx 5 \times 10^{-3} - 10^{-1}$.

[10]A technical issue remains in the question of convergence for the series on the right-hand side of Eq. (62).